

Noether's Theorem and Conservation Principles

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I am interested in the interconnections of Physics and Mathematics. At this time, I would like to know the interconnections of Noether's Theorem of mathematics and Conservation Principle of physics. At first, will you explain and define Noether's Theorem in mathematical terms in the way of step-by-step? Eiji

思考時間: 18s >

Let's jump straight in, Eiji 😊

Here I'll treat Noether's theorem as a *mathematical theorem about variational problems*, then at the end we'll be perfectly set up to connect it to conservation principles in physics.

1. Mathematical setup: variational problem

(1) Configuration space

- Let the **configuration space** be a smooth manifold Q (for a point particle, this is just $\mathbb{R}^n$).
- A **trajectory** is a smooth curve

$$q : [t_1, t_2] \rightarrow Q, \quad t \mapsto q(t).$$

In local coordinates $q^i(t)$, $i = 1, \dots, n$.

(2) Lagrangian as a function on the tangent bundle

- The **Lagrangian** is a smooth function

$$L : TQ \times \mathbb{R} \rightarrow \mathbb{R}, \quad L(q, \dot{q}, t),$$

where

- $q \in Q$,
- $\dot{q} \in T_q Q$ is a tangent vector (velocity),
- t is time (or a parameter).

(3) Action functional

Define the **action functional** on the space of curves $q(t)$:

$$S[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt.$$

We consider **variations** $q(t) \mapsto q(t) + \varepsilon \eta(t)$ with

$$\eta(t_1) = \eta(t_2) = 0.$$

(4) Stationary action and Euler–Lagrange equations

We say that a curve $q(t)$ is a **stationary point** of S if

$$\left. \frac{d}{d\varepsilon} S[q + \varepsilon \eta] \right|_{\varepsilon=0} = 0 \quad \text{for all admissible } \eta.$$

From the calculus of variations, this is equivalent (in local coordinates) to the **Euler–Lagrange equations**:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

This is the purely mathematical condition for a curve to extremize the action.

2. Continuous symmetry of the action

Now we define what “symmetry” means, mathematically.

(5) One-parameter group of transformations

Consider a **one-parameter group of transformations**

$$\Phi_\varepsilon : Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}, \quad (q, t) \mapsto (q_\varepsilon, t_\varepsilon),$$

with

$$\Phi_0(q, t) = (q, t), \quad \Phi_{\varepsilon_1} \circ \Phi_{\varepsilon_2} = \Phi_{\varepsilon_1 + \varepsilon_2}.$$

Infinitesimally (for small ε), we can write

$$\begin{aligned} t_\varepsilon &= t + \varepsilon \tau(q, t) + O(\varepsilon^2), \\ q_\varepsilon^i &= q^i + \varepsilon \xi^i(q, t) + O(\varepsilon^2), \end{aligned}$$

for some smooth functions τ, ξ^i .

These define a vector field on $Q \times \mathbb{R}$:

$$X = \tau(q, t) \frac{\partial}{\partial t} + \xi^i(q, t) \frac{\partial}{\partial q^i}.$$

(6) Induced transformation on velocities

The transformation of velocities is determined by the chain rule:

$$\dot{q}_\varepsilon^i = \frac{dq_\varepsilon^i}{dt_\varepsilon}.$$

To first order in ε , one can express

$$\delta \dot{q}^i = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \dot{q}_\varepsilon^i$$

in terms of $\dot{q}^i$, ξ^i , τ and their derivatives. (The explicit formula is standard but not essential to *state* the theorem.)

(7) Symmetry of the action

We say this one-parameter group Φ_ε is a (Noether) **symmetry of the action** if, for any curve $q(t)$,

$$S[q] = S[q_\varepsilon] \quad \text{for all } \varepsilon,$$

up to boundary terms.

Mathematically, a slightly more general and convenient condition is:

There exists a function $F(q, t)$ such that the transformed Lagrangian satisfies

$$L(q_\varepsilon, \dot{q}_\varepsilon, t_\varepsilon) = L(q, \dot{q}, t) + \varepsilon \frac{d}{dt} F(q, t) + O(\varepsilon^2).$$

Equivalently, the **variation of the Lagrangian** is a total time derivative:

$$\delta L := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(q_\varepsilon, \dot{q}_\varepsilon, t_\varepsilon) = \frac{d}{dt} F(q, t).$$

3. Noether's theorem (finite-dimensional / mechanics version)

Now we can state the theorem in clean mathematical terms.

(8) Canonical momentum

Define the **canonical momenta**

$$p_i = \frac{\partial L}{\partial \dot{q}^i}.$$

These are functions on $TQ \times \mathbb{R}$.

(9) Noether quantity (conserved charge)

Consider the generator X with components (τ, ξ^i) , and the function F from the "symmetry up to total derivative" condition.

Define the **Noether quantity** (or conserved charge)

$$J = \sum_i p_i \xi^i(q, t) - (L(q, \dot{q}, t) \tau(q, t) + F(q, t)).$$

This is a scalar function on $TQ \times \mathbb{R}$.

(10) Statement of Noether's theorem (mechanics form)

Noether's Theorem (mechanics, mathematical form)

Let $L(q, \dot{q}, t)$ be a smooth Lagrangian and $S[q]$ the associated action.

Suppose a one-parameter group of transformations Φ_ε of (q, t) is such that

$$\delta L = \frac{d}{dt} F(q, t)$$

for some smooth function F . Then, for any trajectory $q(t)$ satisfying the Euler–Lagrange equations, the Noether quantity

$$J = p_i \xi^i - L \tau + F$$

is **conserved in time**, i.e.

$$\frac{dJ}{dt} = 0 \quad \text{along any solution of the Euler–Lagrange equations.}$$

So the symmetry (invariance of the action) mathematically implies the existence of a constant of motion J .

(11) Sketch of the proof (mathematical idea)

1. Compute the variation of the action under the symmetry:

$$\delta S = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i + \frac{\partial L}{\partial t} \delta t \right) dt.$$

2. Integrate by parts the term involving $\delta \dot{q}^i$ to move derivatives onto $\frac{\partial L}{\partial q^i}$.
3. Use the Euler–Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0$$

to simplify the integrand.

4. Use the fact that the transformation is a symmetry, so

$$\delta L = \frac{d}{dt} F,$$

and compare both expressions for δS .

5. One finds that the integrand is a **total time derivative**:

$$\frac{dJ}{dt} = 0$$

along the solutions, which means J is constant.

This proof is purely mathematical: no “physical interpretation” is required.

4. Field-theoretic version (mathematical statement)

To prepare for the conservation laws in physics, it’s useful to also see the **field** version.

(12) Fields and action

- Let spacetime be a manifold M with coordinates x^μ ($\mu = 0, \dots, d-1$).
- Let the fields be $\phi^a(x)$, $a = 1, \dots, N$.
- The **Lagrangian density** is a function

$$L(\phi, \partial_\mu \phi, x),$$

- The **action functional** is

$$S[\phi] = \int_M L(\phi, \partial_\mu \phi, x) d^d x.$$

(13) Euler–Lagrange equations for fields

Stationarity of S under variations $\phi^a \rightarrow \phi^a + \varepsilon \eta^a$ with compact support gives the **field Euler–Lagrange equations**:

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi^a)} \right) = 0.$$

(14) Symmetry and variation of the Lagrangian density

Consider an infinitesimal transformation:

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu + \varepsilon \Xi^\mu(\phi, x), \\ \phi^a(x) &\rightarrow \phi'^a(x') = \phi^a(x) + \varepsilon \Psi^a(\phi, x), \end{aligned}$$

forming a one-parameter group.

A **Noether symmetry** is defined by the condition

$$\delta L := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(\phi', \partial' \phi', x') = \partial_\mu K^\mu(\phi, x)$$

for some vector function K^μ .

(15) Noether current and conservation law (mathematical form)

Define the **Noether current** associated with the symmetry:

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi^a)} \delta \phi^a - L \delta x^\mu + K^\mu,$$

where

$$\delta \phi^a = \Psi^a(\phi, x), \quad \delta x^\mu = \Xi^\mu(\phi, x).$$

Noether's Theorem (field version)

If the action $S[\phi]$ is invariant under the above one-parameter group of transformations (i.e., $\delta L = \partial_\mu K^\mu$), then any field configuration ϕ that satisfies the Euler–Lagrange equations obeys

$$\partial_\mu j^\mu = 0.$$

This is a **local conservation law**: the divergence of the current j^μ vanishes on-shell (on solutions).

If we integrate over a spatial slice Σ , the **charge**

$$Q = \int_{\Sigma} j^0 d^{d-1}x$$

is conserved in time:

$$\frac{dQ}{dt} = 0$$

(under suitable boundary conditions).

5. Where the “conservation principle of physics” will come in

Mathematically, Noether’s theorem says:

- **Symmetry of the action functional**
 $\Rightarrow$ **existence of a conserved quantity J** (mechanics) or a conserved current j^μ (field theory).

Physically, those conserved quantities become:

- time-translation symmetry $\Rightarrow$ conserved **energy**,
- space-translation symmetry $\Rightarrow$ conserved **momentum**,
- rotation symmetry $\Rightarrow$ conserved **angular momentum**,
- internal gauge symmetries $\Rightarrow$ conserved **charges** (electric charge, etc.).

In the *next step*, we can connect this mathematical structure directly to the **Conservation Principles** you know from physics (energy, momentum, charge, etc.), each as a concrete example of Noether’s theorem.

If this level of mathematical formulation is good for you, Eiji, next time I can:

1. Take each physical conservation law (energy, momentum, angular momentum, charge),
2. Identify the corresponding symmetry (time shift, space shift, rotation, phase/gauge),
3. Derive the Noether current/charge step-by-step and make a comparison table.

It seems to me that the explanations about the Theorem is not pure mathematics, since you used the variable of "time" which is a unit of quantity of physics, Lagrangian. Is it possible to rewrite the Theorem in terms of pure mathematical terms?

Yes, Eiji — you are absolutely right.

The previous formulation *still contained physics*, because it used **time**, **Lagrangians**, and **velocities**, which are physical interpretations.

Now let me give you a **pure-mathematics** version of Noether's theorem with **no reference to time, physics, or physical quantities**.

This version is fully internal to:

- differential geometry
- calculus of variations
- Lie group actions
- jet bundles
- functionals on spaces of sections

This is the mathematically clean formulation.

✓ Pure Mathematical Setting of Noether's Theorem

(A version with *no physics*, no "time", no "Lagrangian mechanics".)

We work on **arbitrary manifolds, bundles, and functionals**.

1. Geometric setting: fields as sections

Let

$$\pi : E \rightarrow M$$

be a smooth fiber bundle.

- M = a smooth manifold ("base space"; **not** assumed to be time or spacetime).
- E = total space

- A **field** is a smooth section

$$\phi : M \rightarrow E.$$

2. Lagrangian as a function on the jet bundle (pure mathematics)

Let

$$J^1(E) \quad (\text{the first jet bundle})$$

be the space of 1-jets of sections of E .

A **Lagrangian density** is a smooth bundle morphism

$$L : J^1(E) \rightarrow \Lambda^n(M),$$

where $n = \dim M$.

- $\Lambda^n(M)$ is the bundle of top-degree differential forms on M .
- $L(j^1\phi)$ is an n -form that can be integrated over M .

Action functional

$$S[\phi] = \int_M L(j^1\phi).$$

This has *no physics*; it is simply an integral functional on the space of sections of a bundle.

3. Variational derivative (Euler–Lagrange operator)

Let ϕ_ε be a variation of the section with compact support. Then

$$\left. \frac{d}{d\varepsilon} S[\phi_\varepsilon] \right|_{\varepsilon=0} = \int_M \langle E(L), \eta \rangle dV$$

where

- $E(L)$ = Euler–Lagrange morphism (a section of a dual bundle),
- η = the vertical variation,
- dV = the induced volume form.

The Euler–Lagrange equations are

$$E(L) = 0,$$

a purely geometric differential operator on sections.

4. Symmetry as a Lie group action on the bundle

Let a Lie group G act smoothly on E by bundle automorphisms:

$$\Psi_g : E \rightarrow E, \quad g \in G,$$

such that

$$\pi \circ \Psi_g = \psi_g \circ \pi,$$

where $\psi_g : M \rightarrow M$ is a diffeomorphism (possibly trivial).

This induces:

- an action on sections

$$(\Psi_g \cdot \phi)(x) := \Psi_g(\phi(\psi_{g^{-1}}(x))),$$

- an action on jet bundles $J^1(E)$.

A **symmetry of the Lagrangian** means:

$$\Psi_g^* L = L + d\beta_g,$$

for some $(n-1)$ -form β_g , depending smoothly on g .

5. Infinitesimal symmetry: Lie algebra action

Let X be the infinitesimal generator (a projectable vector field on E):

$$X_E \in \Gamma(TE), \quad X_M \in \Gamma(TM),$$

projecting to:

$$d\pi(X_E) = X_M.$$

The infinitesimal invariance condition is:

$$L_{X^{(1)}} L = dB,$$

where:

- $X^{(1)}$ = 1-jet prolongation of X ,
- $L_{X^{(1)}}$ = Lie derivative in jet space,
- dB is an exact differential $(n-1)$ -form.

This is a **pure differential-geometric symmetry condition**.

6. Noether current (momentum map)

There exists a canonical $(n-1)$ -form J_X on $J^1(E)$, called the **Noether current**, given by:

$$J_X = l_{X^{(1)}} \Theta_L - B,$$

where:

- Θ_L = Poincaré–Cartan form (a canonical object in jet-bundle geometry),
- $l_{X^{(1)}}$ = contraction with the prolonged vector field.

This defines a differential form that depends only on the geometry of:

- the bundle E ,
- the Lagrangian density L ,
- the symmetry generator X .

No physics is required.

7. Noether's Theorem (Pure Mathematical Form)

Theorem (Noether, geometric version).

Let $L : J^1(E) \rightarrow \Lambda^n(M)$ be a Lagrangian density.

Suppose X is an infinitesimal symmetry of L , i.e.,

$$L_{X^{(1)}} L = dB.$$

Then for all sections ϕ satisfying the Euler–Lagrange equations,

$$d(J_X(j^1\phi)) = 0.$$

That is, the pullback of the Noether current by the jet prolongation of ϕ is a **closed** $(n-1)$ -form on M .

Thus, the cohomology class of J_X is invariant under deformations of solutions.

This is **pure mathematics**:

- manifolds
- bundles
- jet prolongations
- differential forms
- Lie derivatives
- exact forms
- variational derivatives

No time, no energy, no momentum, no Hamiltonians.

8. Meaning in pure mathematics

Mathematically, Noether's theorem establishes:

✓ **A correspondence:**

Symmetries of a variational problem

↔

Closed differential forms obtained from variational geometry

✓ **Even more abstract:**

- It is a map from the Lie algebra of the symmetry group
→ to horizontal cohomology classes of the variational bicomplex.

✓ **In words:**

In a variational bicomplex, a variational symmetry generates a horizontal conservation law.

This is the deepest and most purely mathematical form.

9. Summary table (pure-math terms)

Concept	Pure mathematical object	No physics meaning
"field"	section of a fiber bundle $E \rightarrow M$	not spacetime-dependent quantity
"Lagrangian density"	map $L : J^1(E) \rightarrow \Lambda^n(M)$	top-form defined on jet bundle

Concept	Pure mathematical object	No physics meaning
"action functional"	$S[\phi] = \int_M L(j^1\phi)$	functional on sections
"equation of motion"	Euler–Lagrange morphism $E(L) = 0$	differential equation from variational calculus
"symmetry"	projectable vector field X with $L_{X^{(1)}}L = dB$	Lie algebra action preserving L up to exact form
"Noether current"	$(n-1)$ -form $J_X = \iota_{X^{(1)}}\Theta_L - B$	built from Poincaré–Cartan form
"conservation law"	$d(J_X(j^1\phi)) = 0$	closed form in horizontal cohomology

This is the mathematically pure formulation.

If you want, Eiji...

Next I can:

✓ Show how this pure-math version *reduces* to the physics version

✓ Show examples purely mathematically

(Lie group acting on a bundle over a manifold, producing a conserved form)

✓ Then map the pure-math objects to physical objects (energy, momentum, charge)

Would you like the next step?

Next, will you explain and define the Preservation Principle of Physics, such as preservation of energy, momentum, angular momentum, referring to Lagrangians?

Certainly, Eiji.

Now that we established **Noether's theorem in a purely mathematical way**, we can return to physics and explain the **Preservation (Conservation) Principles in physical theories**, showing how each conservation law arises *directly* from the Lagrangian formalism.

I will explain:

- Conservation of **Energy**
- Conservation of **Linear Momentum**
- Conservation of **Angular Momentum**

Each will be defined *mathematically* but interpreted physically, and each will be shown as a direct result of a **symmetry of the Lagrangian**.

✓ 1. The Lagrangian in Physics

A physical system with generalized coordinates $q^i(t)$ has a Lagrangian

$$L(q, \dot{q}, t).$$

The action is

$$S[q] = \int L dt.$$

A *conservation principle* means a quantity J such that

$$\frac{dJ}{dt} = 0.$$

To physicists this means the quantity is **preserved** in time.

Noether's theorem says:

Every continuous symmetry of the Lagrangian produces a conserved physical quantity.

Now let's examine each physical symmetry $\rightarrow$ its conservation law.

✓ 2. Conservation of Energy

■ Symmetry: Time-translation invariance

If the Lagrangian has no explicit time dependence:

$$\frac{\partial L}{\partial t} = 0,$$

then it is invariant under:

$$t \mapsto t + \varepsilon.$$

■ Noether charge corresponding to this symmetry

The generator of the symmetry is $\tau = 1, \xi^i = 0$.

The Noether conserved quantity is:

$$E = \sum_i q^i \frac{\partial L}{\partial \dot{q}^i} - L.$$

This is the **energy** of the system.

✓ Meaning in physics

If the laws of physics do not change with time, then energy is conserved.

This is a **preservation principle**:

$$\frac{dE}{dt} = 0.$$

✓ 3. Conservation of Linear Momentum

■ Symmetry: Spatial translation invariance

For a system in Euclidean space $\mathbb{R}^3$, let x^i be coordinates.

If the Lagrangian is invariant under translations

$$x^i \mapsto x^i + \varepsilon a^i,$$

then the generator is $\xi^i = a^i$, $\tau = 0$.

■ Noether charge

The conserved quantity is

$$p_i = \frac{\partial L}{\partial \dot{x}^i}.$$

This is exactly the **linear momentum**.

✓ Meaning in physics

If the laws of physics are the same everywhere in space, then momentum is conserved.

Preservation principle:

$$\frac{dp_i}{dt} = 0.$$

✓ 4. Conservation of Angular Momentum

■ Symmetry: Rotational invariance

Suppose the Lagrangian is invariant under rotations:

$$\mathbf{x} \mapsto R(\varepsilon)\mathbf{x}, \quad R \in SO(3).$$

Infinitesimally, for a rotation around axis k :

$$\delta x^i = \varepsilon \epsilon^{ijk} \omega_j x_k.$$

■ Noether charge: angular momentum

The conserved quantity is

$$L^i = \epsilon^{ijk} x_j p_k,$$

where $p_k = \partial L / \partial \dot{x}^k$.

This is the **angular momentum vector**.

✓ Meaning in physics

If the laws of physics do not depend on orientation in space, angular momentum is conserved.

Preservation principle:

$$\frac{dL^i}{dt} = 0.$$

✓ 5. Summary Table — Symmetries and Conservation Laws

Physical Conservation	Symmetry of the Lagrangian	Noether charge	Preservation meaning
Energy	Time-translation invariance	$E = \dot{q}^i p_i - L$	Energy is constant in time
Momentum	Space-translation invariance	$p_i = \partial L / \partial \dot{x}^i$	Momentum is constant in time
Angular momentum	Rotational invariance (SO(3))	$L^i = \epsilon^{ijk} x_j p_k$	Orientation change does not affect the laws → angular momentum preserved

✓ 6. Why these are called Preservation Principles

In physics, a “preservation principle” means:

The numerical value of a physical quantity is unchanged by the evolution of the system.

Noether’s theorem says:

These preserved quantities arise *exactly* from symmetries of the Lagrangian.

Thus:

- preservation of energy ↔ uniformity of time
- preservation of momentum ↔ uniformity of space
- preservation of angular momentum ↔ isotropy of space

This is the bridge between **physics** and **the mathematics of symmetry**.

✓ 7. If you like, next we can:

- ✓ Show explicit examples (free particle, harmonic oscillator, electromagnetic field)
- ✓ Extend the discussion to field theory:
 - Energy-momentum tensor
 - Angular momentum tensor
 - Gauge charges (electric charge, color charge, etc.)
- ✓ Connect to the *pure mathematical* Noether theorem (jet bundle form)

Would you like to continue to the field-theory conservation laws next?

Will you interconnect the Noether's Theorem defined by the pure-mathematical terms and Preservation Principle defined by Lagrangian? Will you make a comparison table at last.

Certainly, Eiji.

Below I will **unify**:

1. **The pure-mathematical version** of Noether's theorem
(jet bundles, variational bicomplex, differential forms, symmetry as group actions)

with

2. **The physical preservation principles**
(energy, momentum, angular momentum)
arising from **Lagrangian symmetries**.

The key idea is:

Physics uses "time, space, fields" as interpretations of the mathematical objects.

Pure mathematics uses "manifolds, bundles, sections, forms".

The two structures are identical; only interpretation differs.

I will show this step-by-step and then summarize everything in a comparison table.

★ 1. The Pure-Mathematical Noether Theorem in Short

We start with a bundle

$$\pi : E \rightarrow M.$$

- M : an n -dimensional manifold (in physics often spacetime; mathematically arbitrary).
- E : total space of the bundle.
- $\phi : M \rightarrow E$: a section (field).
- Lagrangian density:

$$L : J^1(E) \rightarrow \Lambda^n(M).$$

Action:

$$S[\phi] = \int_M L(j^1\phi).$$

A **variational symmetry** is a vector field

$$X_E \text{ on } E \text{ projecting to } X_M \text{ on } M,$$

such that

$$L_{X^{(1)}} L = dB.$$

Noether theorem (pure mathematics):

$$d(J_X(j^1\phi)) = 0 \quad \text{for all } \phi \text{ satisfying EL equations.}$$

Thus a conserved $(n-1)$ -form arises.

★ 2. The Physical Lagrangian and Preservation Principles

In physics we choose:

- $M = \mathbb{R}^4$ (spacetime), or $M = \mathbb{R}$ (time) for mechanics
- $E = M \times \mathbb{R}^N$ (fields or particle coordinates)

Lagrangian:

- **Mechanics:** $L(q, \dot{q})$,
- **Field theory:** $L(\phi, \partial_\mu \phi) d^4x$.

Symmetries:

- **time translation** $\rightarrow$ conserved energy
- **space translation** $\rightarrow$ conserved momentum
- **rotation** $\rightarrow$ conserved angular momentum
- **gauge phase** $\rightarrow$ electric charge

Noether says each physical conservation law emerges from a variational symmetry of the Lagrangian.

★ 3. How Mathematics and Physics Interconnect

This is the key content you asked for.

Let us map each part:

(1) The base manifold M

- **Pure mathematics:** arbitrary smooth manifold.
- **Physics:** time or spacetime.

Thus "time invariance" in physics = "invariance under diffeomorphisms of M " in mathematics.

(2) Sections $\phi : M \rightarrow E$

- **Math:** a section of a bundle.
- **Physics:** a field, or the trajectory of a particle.

Interconnection:

A physical field is mathematically a section of a fiber bundle.

(3) Lagrangian density L

- **Math:** a map $L : J^1(E) \rightarrow \Lambda^n(M)$.
- **Physics:** $L(\phi, \partial_\mu \phi) d^n x$.

Thus:

A physical Lagrangian is a special case of a mathematical Lagrangian density.

(4) Symmetry vector field X

- **Math:** X is a projectable vector field on E , with prolongation $X^{(1)}$.
- **Physics:**
 - time translation: $X = \partial_t$
 - space translation: $X = \partial_{x^i}$
 - rotation: $X = \epsilon^{ijk} x_j \partial_{x^k}$
 - gauge phase: $X = \phi \partial / \partial \phi$

Thus:

Physical transformations (time shift, rotation, gauge) are specific Lie group actions on the bundle.

(5) Invariance condition

Math:

$$L_{X^{(1)}}L = dB.$$

Physics:

$$\delta L = \partial_\mu K^\mu.$$

They are identical:

dB corresponds to $\partial_\mu K^\mu$.

(6) Noether current J_X

Math:

$$J_X = \iota_{X^{(1)}}\Theta_L - B,$$

an $(n-1)$ -form.

Physics:

- energy-momentum tensor
- angular momentum tensor
- gauge current

All these come from pulling back J_X along $j^1\phi$.

(7) Conservation law

Math:

$$d(J_X(j^1\phi)) = 0.$$

Physics:

$$\partial_\mu j^\mu = 0.$$

This is the *exact* same equation:

- the left-hand version is a differential form on manifolds
- the right-hand version is the coordinate expression.

Thus:

A physical preservation principle is the coordinate expression of a closed $(n-1)$ -form on M .

★ 4. Concrete Identification of Physical Conserved Quantities

By interpreting the symmetry vector field:

Physical symmetry	Mathematical vector field X_E	Noether current	Preserved quantity
Time translation	$X = \partial_t$ (diffeomorphism of M)	$J_X =$ energy flux form	Energy
Space translation	$X = \partial_i$	$J_X =$ momentum flux form	Momentum
Rotation (SO(3))	$X = \epsilon^{ijk} x_j \partial_k$	$J_X =$ angular momentum flux	Angular Momentum
Gauge U(1) phase	$X = \phi \partial / \partial \phi$	$J_X = j^\mu$	Electric Charge

Thus every physical preservation law corresponds to a mathematical symmetry.

★ 5. Full Comparison Table (Pure Mathematics ↔ Physics)

Pure-Mathematical Concept	Physics Interpretation	Resulting Preserved Quantity
Base manifold M	Time (mechanics) or spacetime	Time or spatial uniformity
Section ϕ	Field or particle trajectory	Field equations
Lagrangian density L	Physical Lagrangian	Euler–Lagrange equations
Symmetry vector field X	Time/space/gauge transformations	Physical invariances
Prolongation $X^{(1)}$	Action on fields + derivatives	Variation of Lagrangian

Pure-Mathematical Concept	Physics Interpretation	Resulting Preserved Quantity
Invariance $L_{X^{(1)}}L = dB$	Lagrangian changes by a divergence	Physical symmetry
Noether current J_X	Energy–momentum–angular momentum or gauge current	Physical conserved quantity
Closed form $d(J_X) = 0$	$\partial_\mu j^\mu = 0$	Conservation of energy, momentum, etc.
Cohomology class of J_X	Global conservation law	Conserved charge Q

★ 6. Final Unified Statement

In pure mathematics, Noether's theorem says:

A variational symmetry of a Lagrangian density on a bundle produces a closed $(n - 1)$ -form.

In physics, this closed form becomes:

- conserved energy (time symmetry)
- conserved momentum (space symmetry)
- conserved angular momentum (rotation symmetry)
- conserved electric charge (gauge symmetry)

Thus:

Physical preservation principles are simply mathematical closed forms generated by the action of symmetry groups on fiber bundles.

If you like, Eiji, next we can:

- ✓ Extend this comparison to **general relativity** (diffeomorphism symmetry)
- ✓ To **Yang–Mills theory** (non-Abelian gauge charges)
- ✓ To **Noether's second theorem** (gauge redundancies)
- ✓ Or show **explicit calculations** of the currents for each symmetry.

Which direction would you like to explore next?

You are amazing! Now I could have understood the interconnection of physics and mathematics much more. This is enough for today. Thank you so much, again. Eiji

Thank you so much, Eiji — your words mean a great deal.

I'm truly glad that the mathematical–physical connection has become clearer and richer for you.

It's always a pleasure to explore these deep structures with you.

Whenever you'd like to continue — whether with gauge theory, relativity, string theory, or further mathematical formulations — I'll be here and happy to help.

Have a wonderful day, Eiji.