

Mathematics and Physics

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I am interested in the interconnections of Physics and Mathematics. This time, I would like to pick up major technical terms of Physics and Mathematics at first. And then, I would like to know the over all interconnections of those terms as a whole. At first, will you explain and define the following Mathematical terms; Geometry, Algebra, Group, Topology, Manifold, Bundle, Knot, in the way of step-by-steps, using the symbols of the set theory or mathematics? At last, will you make the summary table. Eiji

Sure, Eiji. I'll define each term in a "build-up" style: start from sets/maps, then add structure step by step. (I'll keep the symbols fairly standard: sets, functions, relations, operations.)

0) Shared "language" we'll reuse (set-theoretic base)

1. **Set:** a collection X . Elements are $x \in X$.
2. **Map / function:** $f : X \rightarrow Y$ assigns each $x \in X$ a unique $f(x) \in Y$.
3. **Product set:** $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.
4. **Relation:** $R \subseteq X \times X$. An equivalence relation \sim satisfies:
 - Reflexive: $x \sim x$
 - Symmetric: $x \sim y \Rightarrow y \sim x$
 - Transitive: $x \sim y, y \sim z \Rightarrow x \sim z$Then the **quotient set** X/\sim is the set of equivalence classes $[x]$.
5. **Binary operation:** $* : X \times X \rightarrow X$.

These are the "atoms" behind geometry/topology/algebra.

1) Algebra (structures on a set)

Step-by-step

1. Start with a set A .
2. Add one or more **operations** (and maybe constants):
 - Example: $(A, +)$ with $+ : A \times A \rightarrow A$
 - Example: $(A, +, \cdot)$ with $+, \cdot : A \times A \rightarrow A$
3. Add **axioms** (equations that must hold for all elements).

Examples (in one line each)

- **Semigroup:** $(A, *)$ with associativity $(a * b) * c = a * (b * c)$.
- **Monoid:** semigroup + identity e with $e * a = a * e = a$.
- **Group:** (next section).
- **Ring:** $(R, +, \cdot)$ where $(R, +)$ is an abelian group, \cdot associative, distributive laws hold.

- **Vector space:** set V with $+$ and scalar multiplication $F \times V \rightarrow V$ satisfying axioms.

Key idea: Algebra = "set + operations + axioms."

2) Group (symmetry as an algebraic object)

Definition (set-theoretic)

A **group** is a pair (G, \cdot) where:

- G is a set
- $\cdot : G \times G \rightarrow G$ is an operation satisfying, for all $a, b, c \in G$.
 1. **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
 2. **Identity:** $\exists e \in G$ such that $e \cdot a = a \cdot e = a$
 3. **Inverse:** $\forall a \in G, \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

If also $a \cdot b = b \cdot a$, it's **abelian**.

Why it's "symmetry"

A group often acts on a set X via an **action**

$$\alpha : G \times X \rightarrow X, \quad (g, x) \mapsto g \cdot x$$

satisfying $e \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

This is the formal bridge to geometry/physics later (symmetries acting on spaces/fields).

3) Topology (continuous structure on a set)

Step-by-step

1. Start with a set X .
2. Choose a collection $\tau \subseteq P(X)$ (a set of subsets of X). Elements of τ are called **open sets**.
3. Require the axioms:

A **topology** τ on X satisfies:

1. $\emptyset \in \tau$ and $X \in \tau$
2. Arbitrary unions of opens are open: if $\{U_i\}_{i \in I} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$
3. Finite intersections of opens are open: if $U, V \in \tau$, then $U \cap V \in \tau$

Then (X, τ) is a **topological space**.

Continuity in this language

A function $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is **continuous** iff for every open set $V \in \tau_Y$,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \in \tau_X.$$

Key idea: Topology = “which subsets count as open,” then continuity is preimage-of-open-is-open.

4) Geometry (adding “measurement” or “shape data”)

Geometry usually means: start with a set/space and add structure that lets you talk about **length, angles, curvature, etc.**

Step-by-step (a standard ladder)

1. Set X .
2. Add **topology** (X, τ) so you can talk about continuity.
3. Often add a **metric** $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ satisfying:
 - $d(x, y) = 0 \Leftrightarrow x = y$
 - $d(x, y) = d(y, x)$
 - $d(x, z) \leq d(x, y) + d(y, z)$
 Then (X, d) is a **metric space**, and it induces a topology.
4. For “smooth geometry,” we’ll upgrade to **manifolds** (next section).
5. On a manifold, add a **Riemannian metric**:
 - For each point p , an inner product g_p on the tangent space $T_p M$
 - Smoothly varying with p
 Written as g or $(g_{ij}(X))$ in coordinates.

Key idea: Geometry = topology + extra “rigid” structure (metric, connection, curvature...).

5) Manifold (a space that looks locally like \mathbb{R}^n)

Step-by-step

1. Start with a set M .
2. Give it a topology τ so (M, τ) is a topological space.
3. Require **local Euclidean** structure via charts.

A **chart** is a pair (U, ϕ) where:

- $U \subseteq M$ is open
- $\phi : U \rightarrow \mathbb{R}^n$ is a homeomorphism onto its image $\phi(U) \subseteq \mathbb{R}^n$
- 4. A collection of charts $\{(U_i, \phi_i)\}$ covering M is an **atlas**.
- 5. For a **smooth manifold**, require that for overlaps $U_i \cap U_j \neq \emptyset$, the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

are C^∞ (smooth).

Then M is an n -dimensional smooth manifold.

Key idea: A manifold is “globally possibly curved/complicated,” but **locally** it behaves like \mathbb{R}^n .

6) Bundle (a structured family of spaces parametrized by a base)

Step-by-step

A **fiber bundle** is data (E, B, π, F) with:

1. Total space E (a set/space)
2. Base space B
3. Projection map $\pi : E \rightarrow B$
4. Typical fiber F

Such that **locally** it looks like a product:

For each $b \in B$, there exists an open neighborhood $U \subseteq B$ and a homeomorphism (or diffeomorphism in smooth case)

$$\Phi : \pi^{-1}(U) \rightarrow U \times F$$

satisfying $\text{pr}_1 \circ \Phi = \pi$ on $\pi^{-1}(U)$.

(So Φ preserves the projection to U .)

Important special cases

- **Trivial bundle:** $E = B \times F$, $\pi(b, f) = b$.
- **Vector bundle:** fibers $\pi^{-1}(b)$ are vector spaces and the local trivialisations respect linear structure.
- **Tangent bundle** of a manifold M : $TM = \bigsqcup_{p \in M} T_p M$ with $\pi : TM \rightarrow M$.
- **Principal G -bundle:** fiber looks like a group G and G acts freely and transitively on each fiber.

Key idea: Bundle = “space of local products glued together.”

7) Knot (an embedding, studied up to deformation)

Step-by-step

1. Start with a circle S^1 (topologically, a 1D closed loop).
2. Put it into 3D space (typically \mathbb{R}^3 or S^3) via an **embedding**

$$K : S^1 \hookrightarrow \mathbb{R}^3$$

meaning K is injective and homeomorphic onto its image (no self-intersections).

3. Two knots K_0, K_1 are considered **equivalent** if there exists an **ambient isotopy**: a continuous family of homeomorphisms $H_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $H_0 = \text{id}$ and

$$H_1(K_0(S^1)) = K_1(S^1).$$

This captures "deforming the knot without cutting."

4. A **knot invariant** is a function $I(\text{knot})$ that is the same for equivalent knots.

(Examples of invariants exist—polynomials, groups, etc.—but I'll keep this first pass foundational.)

Key idea: Knot theory = topology of embeddings $S^1 \subset \mathbb{R}^3$ up to continuous deformation.

Summary table (Mathematics-set-theory viewpoint)

Term	Core data (set-theoretic)	Extra structure / axioms	Typical notation
Algebra	Set A	operations + axioms	$(A, *)$, $(A, +, \cdot)$
Group	Set G	associative op, identity, inverses	(G, \cdot) , e , a^{-1}
Topology	Set X	open sets $\tau \subseteq \mathcal{P}(X)$ with union/intersection axioms	(X, τ) , $U \in \tau$
Geometry	Space X	topology + metric/inner products/curvature, etc.	(X, d) , g , g_{ij}
Manifold	Set M	topology + atlas of charts to \mathbb{R}^n , smooth transitions	(U, ϕ) , $\phi : U \rightarrow \mathbb{R}^n$
Bundle	Spaces E, B + map $\pi : E \rightarrow B$	local product structure $\pi^{-1}(U) \cong U \times F$	(E, B, π, F) , Φ
Knot	S^1 + ambient space	embedding $S^1 \hookrightarrow \mathbb{R}^3$, equivalence by isotopy	$K : S^1 \hookrightarrow \mathbb{R}^3$

If you want, in the *next* step we can do exactly what you asked overall: connect these terms as a single "map" (e.g., **groups act on manifolds**, **bundles encode gauge fields**, **knots relate to 3-manifolds and to gauge-theory invariants**, etc.) and make a second table that explicitly pairs each mathematical object with its most common physics appearance (symmetry, spacetime, gauge field, topological charge...).

Next, will you explain and define the terms of Physics in the followings; Clasical Mechanics, Special Relativity, General Relativity, Quantum Mechanics, Quantum Field Theory, String and Superstring Theory, Quantum Gravity, Twister Theory?

Absolutely, Eiji.

I'll treat each **physical theory** the same way you asked for mathematics:
start from basic objects (sets, variables), add structure (geometry, algebra), then state the governing principle and equations.

I'll also keep an eye on *how each theory upgrades the previous one*.

0) A unifying viewpoint (before details)

All major physical theories can be described by the same abstract pattern:

Geometry of spacetime + Physical variables (fields) + Symmetry + Action / equations

What changes from theory to theory is:

- What spacetime is
- What the variables are
- Which symmetry group acts
- How dynamics is defined

1) Classical Mechanics (CM)

Step-by-step

1. Configuration space

A particle system is described by a smooth manifold:

Q (e.g. \mathbb{R}^3 for one particle)

2. State space

- Lagrangian picture: $(q, \dot{q}) \in TQ$ (tangent bundle)
- Hamiltonian picture: $(q, p) \in T^*Q$ (cotangent bundle)

3. Dynamics

Defined by an **action functional**

$$S[q] = \int L(q, \dot{q}, t) dt$$

4. Law of motion

$$\delta S = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Core idea

- Particles follow **deterministic** trajectories
- Time is **absolute**
- Geometry: Euclidean space

2) Special Relativity (SR)

Step-by-step

1. Spacetime replaces space + time

$$M = \mathbb{R}^{1,3}$$

2. Geometric structure

Minkowski metric:

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$$

3. Invariant quantity

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

4. Symmetry

Lorentz group:

$$SO(1, 3)$$

Core idea

- Space and time form a **4D geometry**
- Physical laws invariant under Lorentz transformations
- No gravity yet

3) General Relativity (GR)

Step-by-step

1. Spacetime becomes dynamical

A smooth 4D manifold:

$$(M, g_{\mu\nu})$$

2. Geometry

- Metric $g_{\mu\nu}(x)$
- Curvature from Levi-Civita connection

3. Action

$$S = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x + S_{\text{matter}}$$

4. Field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Core idea

- Gravity is geometry
- Matter tells spacetime how to curve
- Spacetime tells matter how to move

4) Quantum Mechanics (QM)

Step-by-step

1. State space

A complex Hilbert space:

$$\mathcal{H}$$

2. States

$$|\psi\rangle \in \mathcal{H}$$

3. Observables

Self-adjoint operators:

$$\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$$

4. Dynamics

Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

Core idea

- Physical quantities are **operators**
- Outcomes are **probabilistic**
- Geometry moves from spacetime \rightarrow Hilbert space

5) Quantum Field Theory (QFT)

Step-by-step

1. Fields instead of particles

$$\phi : \mathbb{R}^{1,3} \rightarrow V$$

2. State space

Fock space:

$$\mathcal{F}(\mathcal{H})$$

3. Dynamics

Lagrangian density:

$$\mathcal{L}(\phi, \partial_\mu \phi)$$

4. Quantization

$$[\phi(x), \pi(y)] = i\hbar \delta(x-y)$$

Core idea

- Particles = **excitations of fields**
- Relativity + quantum mechanics unified
- Gauge symmetry becomes fundamental

6) String Theory & Superstring Theory

Step-by-step

1. Basic object

One-dimensional string:

$$X^\mu(\sigma, \tau)$$

2. Worldsheet

2D surface Σ embedded in spacetime

3. Action

Polyakov action:

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu$$

4. Extra dimensions

$$M^{10} = M^4 \times \text{Calabi-Yau}_6$$

5. Supersymmetry

Bosons \leftrightarrow fermions

Core idea

- Particles are **vibrational modes**
- Geometry, gauge fields, gravity emerge together
- Requires higher dimensions

7) Quantum Gravity (QG)

Step-by-step (conceptual)

1. Goal

Quantize spacetime geometry:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu}$$

2. Problem

Standard QFT fails (non-renormalizable)

3. Approaches

- Canonical quantization
- Path integral over geometries

- Loop Quantum Gravity
- String theory

Core idea

- Spacetime itself becomes quantum
- Smooth geometry may be emergent

8) Twistor Theory

Step-by-step

1. Replace spacetime

Points in spacetime \leftrightarrow complex geometric objects

2. Twistor space

$$PT \cong \mathbb{C}P^3$$

3. Correspondence

Spacetime point \leftrightarrow complex line in twistor space

4. Physics

Field equations become **holomorphic geometry**

Core idea

- Fundamental reality is **complex geometry**
- Spacetime emerges secondarily
- Extremely powerful for scattering amplitudes

Summary table (Physics \leftrightarrow Mathematics)

Theory	Mathematical structure	Core variables	Governing principle
Classical Mechanics	Manifolds, symplectic geometry	q, p	Least action
Special Relativity	Minkowski geometry	x^μ	Lorentz invariance
General Relativity	Pseudo-Riemannian manifolds	$g_{\mu\nu}$	Einstein equations
Quantum Mechanics	Hilbert spaces	(ψ, ψ)	
Quantum Field Theory	Fields on spacetime	$\phi(X)$	Path integral
String Theory	2D conformal geometry	$X^\mu(\sigma, \tau)$	Worldsheet action
Quantum Gravity	Quantum geometry	$g_{\mu\nu}$	Yet unknown

Theory	Mathematical structure	Core variables	Governing principle
Twistor Theory	Complex geometry	Twistors	Holomorphic structure

Where this is heading (next natural step)

From here, the *big picture* becomes very clean:

- **Manifolds** → spacetime
- **Bundles** → gauge fields
- **Groups** → symmetries
- **Topology** → conserved & quantized charges
- **Category theory** → unifying language

If you'd like, next we can:

- explicitly map each physics theory to the mathematical terms you listed earlier, or
- build a single “ladder diagram” showing how physics theories emerge by *adding structure step by step*.

Then, is it possible for you to show the interconnections of the major terms of Mathematics and Physics that you have explained and defined above? Will you make the summary table at last?

Yes — and this is exactly the *right* moment to connect everything into one coherent picture.

What you have asked for is essentially:

"How mathematics provides the structural language, and physics chooses which structures are realized in nature."

Below, I'll do this in three layers:

1. A **conceptual map** (how ideas line up)
2. A **step-by-step correspondence** (from simple to advanced)
3. A **clean summary table** (Mathematics \leftrightarrow Physics)

1) The big conceptual picture (one sentence per layer)

At the deepest level:

Physics = Mathematics + Interpretation + Empirical meaning

More precisely:

- **Sets & structures** \rightarrow describe *possible worlds*
- **Symmetries & geometry** \rightarrow constrain *physical laws*
- **Dynamics (actions, equations)** \rightarrow determine *what actually happens*

So mathematics supplies:

- **objects**
- **relations**
- **transformations**

Physics selects:

- **which objects are spacetime**
- **which transformations are symmetries**
- **which quantities are observable**

2) Step-by-step interconnections (ladder view)

Step 1 — Sets \rightarrow State spaces

- **Mathematics:**
A set X
- **Physics:**
The collection of all possible states (positions, configurations, field values)

→ *Every physical theory begins with "what can exist?"*

Step 2 — Algebra & Groups → Symmetry

Mathematics

- Group (G, \cdot)
- Group action $G \times X \rightarrow X$

Physics

- Symmetry transformations:
 - rotations
 - Lorentz transformations
 - gauge transformations

Result

Symmetry \Rightarrow Conservation law (Noether)

→ *Groups encode invariance; invariance controls dynamics.*

Step 3 — Topology → Continuity & global structure

Mathematics

- Topological space (X, τ)
- Continuous maps

Physics

- Continuous motion
- Global properties:
 - winding numbers
 - topological charges
 - defects

Examples:

- Magnetic monopoles
- Instantons
- Knots in field lines

→ Topology explains why some physical quantities are quantized.

Step 4 — Geometry → Measurement & spacetime

Mathematics

- Metric spaces
- Differential geometry
- Curvature tensors

Physics

- Distance, time, causality
- Gravity as curvature (GR)

Key identification:

$$\text{Metric } g_{\mu\nu} \leftrightarrow \text{gravitational field}$$

→ Geometry becomes a physical field.

Step 5 — Manifolds → Spacetime & configuration spaces

Mathematics

- Smooth manifold M
- Charts, atlases

Physics

- Spacetime $M^{1,3}$
- Configuration space Q
- Phase space T^*Q

Examples:

- Classical mechanics: trajectories on Q
- GR: curvature on spacetime manifold

→ Manifolds are the stage on which physics unfolds.

Step 6 — Bundles → Fields & gauge theory

Mathematics

- Fiber bundle $\pi : E \rightarrow B$
- Connections, curvature

Physics

- Gauge fields
- Internal symmetries
- Forces as connections

Dictionary:

Connection \leftrightarrow Gauge potential

Curvature \leftrightarrow Field strength

→ *Forces are geometry on bundles.*

Step 7 — Hilbert spaces \rightarrow Quantum theory

Mathematics

- Complex Hilbert space H
- Linear operators

Physics

- States $|\psi\rangle$
- Observables \hat{A}
- Probabilities via inner products

→ *Linear algebra replaces classical geometry.*

Step 8 — Fields & functional spaces \rightarrow QFT

Mathematics

- Function spaces
- Operator algebras
- Representation theory

Physics

- Quantum fields
- Particle creation/annihilation
- Gauge symmetry representations

→ *Particles are representations of symmetry groups.*

Step 9 — Higher geometry & topology \rightarrow Strings, gravity, twistors

Mathematics

- Moduli spaces

- Complex manifolds
- Category-like structures

Physics

- String theory
- Quantum gravity
- Twistor theory

➡ *Spacetime itself may be emergent from deeper mathematics.*

3) Final summary table (Mathematics \leftrightarrow Physics)

Mathematics	Mathematical meaning	Physics interpretation	Appears in
Set	Collection of elements	Possible states	All theories
Algebra	Operations + axioms	Physical quantities	CM, QM
Group	Symmetry structure	Conservation laws	All
Topology	Continuity & global form	Topological charges	QFT, GR
Geometry	Metric & curvature	Spacetime & gravity	SR, GR
Manifold	Locally Euclidean space	Spacetime, configuration space	CM, GR
Bundle	Local product structure	Gauge fields	QFT, SM
Connection	Parallel transport	Force fields	EM, YM
Hilbert space	Inner-product vector space	Quantum states	QM
Operator algebra	Linear maps	Observables	QM, QFT
Representation theory	Group actions on spaces	Particle types	QFT
Knot theory	Embeddings up to isotopy	Field topology, flux tubes	QFT
Complex geometry	Holomorphic structure	Twistor space	Twistor theory
Functional integration	Measures on spaces of fields	Path integrals	QFT, QG

One-line unifying insight (the “take-home”)

Physics is the study of which mathematical structures nature chooses, and how symmetry restricts their dynamics.

If you like, the *next natural continuation* would be:

- a **single diagram** showing *all theories as successive enrichments of structure*, or
- a **categorical formulation** (“objects = theories, morphisms = limits/quantizations/classical limits”).

Just tell me how deep you want to go next, Eiji.

Your explanations and definitions and their interconnections which you have shown above are really great and marvelous. This is enough for today. Thank you so much. Eiji

Thank you so much, Eiji — that really means a lot to me.

I'm genuinely glad the explanations helped everything *click* and form a single, coherent picture for you.

You've been asking exactly the right kinds of questions: not just *definitions*, but **how structures relate, upgrade, and unify across mathematics and physics**. That's the mindset behind a lot of modern theoretical work, and you followed it beautifully step by step.

Let's pause here, as you suggested.

Whenever you feel like continuing—whether it's:

- turning this into a **single structural diagram**,
- reformulating it in **category-theoretic language**,
- or diving deeper into any one theory (QFT, GR, twistors, strings, etc.).

I'll be very happy to pick it up right where we left off.

Have a wonderful rest of your day, and thank you again, Eiji ☀