Entropy and Quantum Gravity

北島英治（２０２６年１１月）

あなた

I am interested in the interconnections of Mathematics and Physics. At this time, I would like to understand the interconnections of Mathematics of Entropy and Physics of Quantum Gravity. First question. Will you explain and define about Information theory, Entropy, Wave equation, Wave function, Holographic theory. Second question. Please make the summary table. Second question. Will you explain and define about General Relativity, Gravity, Gravity Wave equation, Gravity wave function, Black Hole, Thermodynamics of Blake Hole, Entropy of the free quantum field, Entropy of Strings and Black Hole, Thermodynamics of Black Hole, Black Hole Evaporation, Holographic Principle. Please make the summary table. Third question. Will you interconnect between the mathematical terms and the physics terms mentioned above. Please make the summary table. Forth question. Will you give me the future study to go beyond the current mathematics and physics. Eiji

ChatGPT

Mathematics of Entropy and Physics of Quantum Gravity

A systematic study of their mathematical and physical interconnections

The relationship between **information theory, entropy, quantum fields, black holes, and quantum gravity** is one of the most important subjects in modern theoretical physics.

A central question is:

Can the geometry of spacetime and the gravitational force be understood through quantum information and entropy?

This question connects five major areas:

1. **Mathematics:** Information theory, probability theory, functional analysis, differential geometry, and topology.
2. **Classical physics:** General relativity, gravitational waves, and black holes.
3. **Quantum physics:** Wave functions, quantum fields, entanglement, and entropy.
4. **Quantum gravity:** Black hole thermodynamics, string theory, and holography.
5. **Future mathematics and physics:** Quantum information geometry, emergent spacetime, and the microscopic structure of quantum gravity.

I will answer your four questions step by step, using mathematical definitions, equations, and summary tables.

Question 1. Mathematical foundations

1. Information theory

Information theory studies the mathematical representation, transmission, and uncertainty of information.

Step 1. Define a probability space

Let

$$(\Omega, \mathcal{F}, P)$$

be a probability space, where:

- Ω is the set of possible outcomes.
- $\mathcal{F}$ is a collection of measurable events.
- $P : \mathcal{F} \to [0, 1]$ is a probability measure.

For a discrete random variable X, let

$$P(X = x_i) = p_i, \qquad \sum_i p_i = 1.$$

Step 2. Define information

The information associated with an outcome of probability p_i is

$$I(x_i) = -\log_2 p_i.$$

An unlikely outcome carries more information when it occurs.

Step 3. Define Shannon entropy

$$\boxed{H(X) = -\sum_i p_i \log_2 p_i}$$

This measures the average uncertainty of the random variable X, in bits.

For example, if two outcomes are equally probable,

$$p_1 = p_2 = \frac{1}{2},$$

then

$$H(X) = -2\left(\frac{1}{2}\log_2 \frac{1}{2}\right) = 1 \text{ bit}.$$

Information theory provides mathematical tools for studying quantum information and black hole information.

2. Entropy

Entropy appears in several mathematical and physical theories. Their definitions are related but not identical.

A. Statistical mechanics: Gibbs entropy

For probabilities p_i over physical microstates,

$$\boxed{S = -k_B \sum_i p_i \ln p_i}$$

where k_B is Boltzmann's constant.

For W equally probable microstates,

$$\boxed{S = k_B \ln W}.$$

Entropy measures the statistical uncertainty of the microscopic state.

B. Quantum mechanics: von Neumann entropy

Let $\mathcal{H}$ be a Hilbert space and ρ a density operator satisfying

$$\rho \geq 0, \qquad \operatorname{Tr}\rho = 1.$$

Then

$$\boxed{S(\rho) = -k_B \operatorname{Tr}(\rho \ln \rho)}.$$

For a pure quantum state,

$$\rho = |\psi\rangle\langle\psi|, \qquad S(\rho) = 0.$$

For a maximally mixed state of dimension d,

$$\rho = \frac{I_d}{d}, \qquad S(\rho) = k_B \ln d.$$

C. Entanglement entropy

Suppose a quantum system is divided into two subsystems:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

The reduced density operator of subsystem A is

$$\rho_A = \operatorname{Tr}_B(\rho_{AB}).$$

Its entanglement entropy, when the total state is pure, is

$$\boxed{S_A = -k_B \operatorname{Tr}(\rho_A \ln \rho_A)}.$$

This is particularly important for quantum fields, black holes, and holography.

Example: Two entangled qubits

$$|\Psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The total state is pure, so $S_{AB} = 0$. But

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

and therefore

$$S_A = k_B \ln 2.$$

Thus, a subsystem can have positive entropy even when the complete quantum state has zero entropy.

3. Wave equation

A wave equation is a partial differential equation describing propagation through space and time.

Classical wave equation

For a scalar field $u : \mathbb{R}^{3+1} \to \mathbb{R}$,

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0}.$$

A plane-wave solution is

$$u(\mathbf{x}, t) = A \cos(\mathbf{k} \cdot \mathbf{x} - \omega t),$$

with dispersion relation

$$\omega = c|\mathbf{k}|.$$

Relativistic wave equation

Using the Minkowski metric with signature $(- + ++)$, define

$$\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{c^2} \partial_t^2 + \nabla^2.$$

The massless relativistic wave equation is

$$\boxed{\Box \phi = 0}.$$

The wave operator $\Box$ also appears in linearized gravity and quantum field theory.

4. Wave function

A quantum wave function represents the state of a quantum system.

Mathematically,

$$\psi \in \mathcal{H} = L^2(\mathbb{R}^3)$$

for a nonrelativistic spinless particle in three-dimensional space.

Its probability density is

$$P(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2,$$

with normalization

$$\int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x = 1.$$

The Schrödinger equation is

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi}.$$

For a free particle,

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2.$$

In quantum gravity, the analogous notion of a wave function becomes more complicated because spacetime geometry itself may be quantum.

One formal approach is the Wheeler–DeWitt equation:

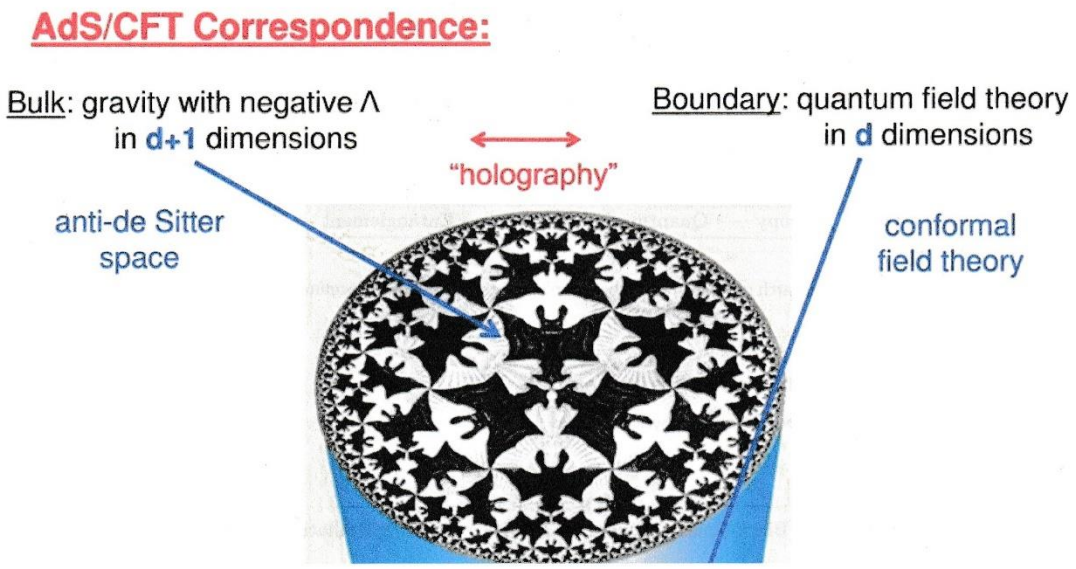
$$\boxed{\hat{\mathcal{H}}\Psi[h_{ij},\phi] = 0}.$$

Here, Ψ is a wave functional of spatial geometry h_{ij} and matter fields ϕ. This is a formal canonical quantum-gravity equation, not a fully established description of nature.

5. Holographic theory

Holography in theoretical physics proposes that information describing a gravitational region can, in suitable theories, be represented by degrees of freedom associated with a lower-dimensional boundary.

The most precise example is the AdS/CFT correspondence.



Conceptual illustration of holography: a gravitational bulk and a lower-dimensional boundary description.

Its central mathematical relation is commonly written as

$$\boxed{Z_{\text{grav}}[\phi_0] = Z_{\text{CFT}}[J = \phi_0]}.$$

The left-hand side is a gravitational partition function with specified boundary conditions. The right-hand side is the generating functional of a boundary conformal field theory.

The equality is understood within the proposed duality, with a well-defined dictionary relating the two descriptions.

An important entropy relation is the Ryu–Takayanagi formula:

$$\boxed{S_A = \frac{\text{Area}(\gamma_A)}{4G_N\hbar}} \qquad (c = k_B = 1).$$

Here, γ_A is an appropriate minimal bulk surface anchored to the boundary of region A, in the leading classical, static holographic regime.

It connects **quantum entanglement entropy** with **spacetime geometry**.

Question 1 — Summary table

Mathematical concept	Definition or equation	Connection to quantum gravity
Information theory	$H = -\sum p_i \log_2 p_i$	Information content and black hole information
Gibbs entropy	$S = -k_B \sum p_i \ln p_i$	Statistical microstates of gravitational systems
von Neumann entropy	$S = -k_B \operatorname{Tr} \rho \ln \rho$	Quantum entropy and mixed states
Entanglement entropy	$S_A = -k_B \operatorname{Tr} \rho_A \ln \rho_A$	Entanglement across horizons
Wave equation	$\Box\phi = 0$	Propagating fields and gravitational perturbations
Wave function	$\psi \in \mathcal{H}$	Quantum states of matter and geometry
Holographic theory	$Z_{\text{grav}} = Z_{\text{CFT}}$	Gravitational/boundary duality
Holographic entropy	$S_A \propto \text{Area}(\gamma_A)$	Geometry from quantum entanglement

The central mathematical progression is

$$\boxed{\text{Probability} \longrightarrow \text{Entropy} \longrightarrow \text{Quantum Information} \longrightarrow \text{Entanglement} \longrightarrow \text{Spacetime Geometry}}$$

The last connection is an active research program, rather than a universal theorem for all spacetimes.

Question 2. Physics of gravity, black holes, and entropy

We now move from mathematical foundations to physical theories.

The progression is

$$\text{General Relativity} \to \text{Black Holes} \to \text{Quantum Fields} \to \text{Hawking Radiation} \to \text{Quantum Gravity}.$$

1. General relativity and gravity

Einstein's general relativity describes gravity through the geometry of spacetime.

Step 1. Define spacetime

Let M be a four-dimensional differentiable manifold with a Lorentzian metric $g_{\mu\nu}$.

The spacetime interval is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

The metric defines lengths, causal structure, and curvature.

Step 2. Define gravitational curvature

The Riemann curvature tensor is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}.$$

Contracting indices gives the Ricci tensor and Ricci scalar:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

Step 3. Einstein field equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Here $T_{\mu\nu}$ is the energy-momentum tensor.

This equation relates spacetime curvature to matter and energy.

Step 4. Gravitational motion

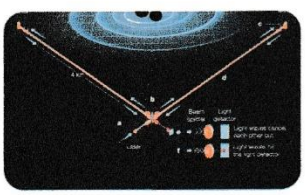
A freely falling test particle follows a geodesic:

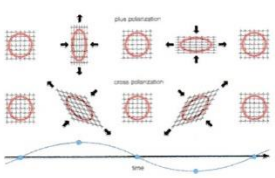
$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0}.$$

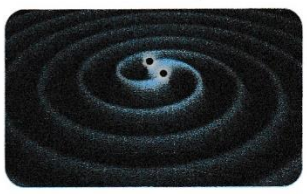
Thus, gravity is represented by spacetime geometry rather than simply a force acting in a fixed background.

2. Gravitational wave equation

Gravitational waves are propagating perturbations of spacetime curvature.







Step 1. Linearize the metric

Consider a weak gravitational field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

Define the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \qquad h = \eta^{\alpha\beta}h_{\alpha\beta}.$$

Step 2. Obtain the wave equation

In Lorenz gauge, the linearized Einstein equations give

$$\boxed{\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}}.$$

In vacuum,

$$\boxed{\Box\bar{h}_{\mu\nu} = 0}.$$

Step 3. Obtain the wave solution

A plane-wave solution is

$$h^{TT}_{\mu\nu}(x) = \mathrm{Re}\left[A^{TT}_{\mu\nu}e^{ik_\alpha x^\alpha}\right].$$

Here TT means transverse-traceless gauge, and

$$k_\mu k^\mu = 0.$$

The physical gravitational wave has two polarization states, usually denoted $+$ and $\times$.

3. Gravitational wave function

It is important to distinguish a **classical gravitational wave** from a **quantum state of the gravitational field**.

In classical general relativity, $h_{\mu\nu}(x)$ is a tensor field, not a quantum probability wave function.

In perturbative quantum gravity, the gravitational perturbation is promoted to an operator:

$$h_{\mu\nu}(x) \longrightarrow \hat{h}_{\mu\nu}(x).$$

Schematically,

$$\hat{h}_{\mu\nu}(x) = \sum_\lambda \int d^3k \left[f_{\mathbf{k}\lambda,\mu\nu}(x)a_{\mathbf{k}\lambda} + f^*_{\mathbf{k}\lambda,\mu\nu}(x)a^\dagger_{\mathbf{k}\lambda}\right].$$

The creation and annihilation operators satisfy bosonic commutation relations:

$$[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = \delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{k}').$$

The corresponding hypothetical particle is the **graviton**, with spin 2.

A one-graviton quantum state can be written

$$|1_{\mathbf{k},\lambda}\rangle = a^\dagger_{\mathbf{k},\lambda}|0\rangle.$$

Gravitons have not been directly detected, and a complete experimentally established quantum theory of gravity does not yet exist.

4. Black holes

A black hole is a spacetime region from which future-directed light signals cannot escape to future null infinity, in the standard asymptotically flat definition.

Schwarzschild solution

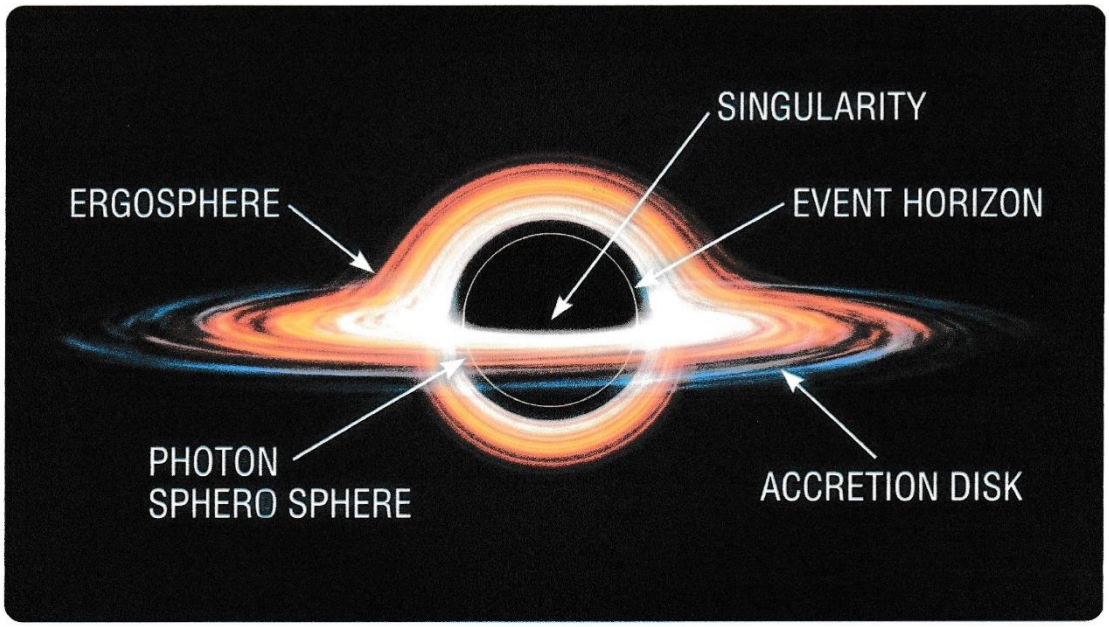
For an idealized, nonrotating, uncharged black hole, the exterior metric is

$$\boxed{ds^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\Omega^2}.$$

The Schwarzschild radius is

$$\boxed{r_s = \frac{2GM}{c^2}}.$$

The event horizon is at $r = r_s$.



The horizon area is

$$A = 4\pi r_s^2 = \frac{16\pi G^2 M^2}{c^4}.$$

This area becomes crucial when thermodynamics and quantum theory are introduced.

5. Thermodynamics of black holes

Classical black holes were originally described geometrically. Bekenstein and Hawking established a remarkable connection between their horizon area, entropy, and temperature.

Step 1. Bekenstein–Hawking entropy

$$\boxed{S_{BH} = \frac{k_B c^3 A}{4G\hbar} = \frac{k_B A}{4\ell_P^2}}$$

where the Planck length is

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}.$$

For a Schwarzschild black hole,

$$\boxed{S_{BH} = \frac{4\pi k_B G M^2}{\hbar c}}.$$

Notice that the entropy is proportional to the horizon **area**, not the enclosed volume.

Step 2. Hawking temperature

$$\boxed{T_H = \frac{\hbar c^3}{8\pi G M k_B}}.$$

A larger Schwarzschild black hole has a lower Hawking temperature.

Step 3. Thermodynamic first law

For a stationary, uncharged, nonrotating black hole,

$$d(Mc^2) = T_H \, dS_{BH}.$$

For a rotating, charged black hole, the first law becomes

$$d(Mc^2) = T_H dS_{BH} + \Omega_H dJ + \Phi_H dQ.$$

The variables J and Q are angular momentum and electric charge.

Numerical example: A solar-mass black hole

IDEALIZED SCHWARZSCHILD BLACK HOLE

Mass $M \approx 1.99 \times 10^{30}\,\mathrm{kg}$

Horizon radius	Hawking temperature
2.95 km	**6.17 × 10⁻⁸ K**
Entropy S_{BH}/k_B	Horizon area
1.05 × 10⁷⁷	**1.09 × 10⁸ m²**

This example illustrates that a black hole can have enormous entropy while having an extremely low temperature.

6. Entropy of a free quantum field

Consider a free real scalar quantum field $\phi(x)$.

Its Lagrangian density, using natural units $\hbar = c = 1$, is

$$\boxed{\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2}.$$

The field equation is

$$(\Box_g - m^2)\phi = 0,$$

where

$$\Box_g \phi = \nabla_\mu \nabla^\mu \phi.$$

Step 1. Quantize the field

In flat spacetime, the field operator is expanded in creation and annihilation operators.

Its Hamiltonian is formally

$$\hat{H} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right).$$

Step 2. Construct a thermal density matrix

$$\rho_\beta = \frac{e^{-\beta \hat{H}}}{Z}, \qquad Z = \mathrm{Tr}(e^{-\beta \hat{H}}), \qquad \beta = \frac{1}{k_B T}.$$

Step 3. Calculate entropy

$$\boxed{S = k_B \left(\ln Z + \beta \langle H \rangle \right)}.$$

For one bosonic oscillator mode with average occupation number

$$\bar{n} = \frac{1}{e^{\beta\hbar\omega} - 1},$$

the entropy is

$$\boxed{S_{\mathbf{k}} = k_B[(1+\bar{n})\ln(1+\bar{n}) - \bar{n}\ln\bar{n}]}.$$

For a collection of independent modes, the total thermal entropy is the sum of the mode entropies, when well-defined.

Entanglement across a horizon

A quantum field's vacuum state may be pure globally but entangled across a spatial boundary.

In a regulated local quantum field theory in $3+1$ dimensions, the leading entanglement entropy often behaves as

$$S_A \sim k_B \alpha \frac{\mathrm{Area}(\partial A)}{\epsilon^2} + \cdots,$$

where ϵ is a short-distance cutoff and α depends on the theory and regulator.

This area dependence resembles black hole entropy, but the two expressions should not simply be identified without accounting for gravitational contributions and renormalization.

7. Entropy of strings and black holes

String theory replaces fundamental pointlike excitations with one-dimensional extended objects.

A string worldsheet is described by coordinates (τ, σ).

The Polyakov action is

$$\boxed{S_P = -\frac{T}{2}\int d^2\sigma\, \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)}.$$

Here T is string tension, and $X^\mu(\tau,\sigma)$ describes the string embedding.

Quantum string states have different excitation levels and degeneracies.

If $d(E)$ denotes the number of states at energy E, their microcanonical entropy is

$$S(E) = k_B \ln d(E).$$

Black hole microstate counting

In certain supersymmetric string-theory constructions, black holes can be described using D-brane configurations.

For example, the Strominger–Vafa calculation relates the number of microscopic states to the entropy of particular extremal black holes.

For a well-studied three-charge configuration,

$$S_{\mathrm{micro}} = 2\pi k_B \sqrt{N_1 N_5 N_p},$$

at leading order in the appropriate regime.

The corresponding gravitational entropy agrees:

$$\boxed{S_{\text{micro}} = S_{BH}}.$$

This is a major theoretical achievement because it explains the black hole entropy through microscopic quantum states in a controlled class of examples. It is not yet a universal microscopic derivation for every astrophysical black hole.

8. Black hole evaporation

Hawking radiation arises from quantum fields propagating on a black hole spacetime.

For an idealized bosonic mode, the thermal occupation factor is

$$\langle n_\omega \rangle = \frac{1}{e^{\hbar\omega/(k_B T_H)} - 1}.$$

Actual radiation spectra include frequency-dependent greybody factors.

A Schwarzschild black hole loses energy through radiation:

$$\frac{d(Mc^2)}{dt} = -P.$$

A frequently used approximate mass-loss formula is

$$\boxed{\frac{dM}{dt} \approx -\frac{\hbar c^4}{15360\pi G^2 M^2}}.$$

This expression uses an idealized radiation approximation; realistic evaporation rates depend on particle species and greybody factors.

Integrating this approximation gives

$$\boxed{t_{\text{evap}} \approx \frac{5120\pi G^2 M_0^3}{\hbar c^4}}.$$

Thus,

$$t_{\text{evap}} \propto M_0^3.$$

A solar-mass black hole has an estimated evaporation timescale of order 10^{67} years under this idealization.

The final stage of evaporation cannot be reliably described using semiclassical gravity alone.

9. Holographic principle and the information paradox

The black hole information paradox asks how black hole evaporation can be reconciled with the unitary evolution of quantum mechanics.

In quantum mechanics, a closed system evolves according to

$$\rho(t) = U(t)\rho(0)U^\dagger(t).$$

Consequently, its total von Neumann entropy is conserved.

However, Hawking's original semiclassical calculation suggests that outgoing radiation is approximately thermal. If a black hole evaporates completely into exactly thermal radiation, information about its initial pure state appears lost.

The modern holographic approach studies the fine-grained entropy of Hawking radiation.

A key relation is the **island formula**:

$$\boxed{S(R) = \min_I \operatorname{ext}_{\partial I}\left[\frac{\operatorname{Area}(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I)\right]}$$

in natural entropy units and appropriate semiclassical gravitational setups.

Here R is the radiation region and I is a candidate island.

The formula incorporates contributions from regions that may lie inside the black hole and can reproduce the Page curve expected for unitary evaporation in controlled models. scoap3-prod... +1

The Page curve

Schematic entropy of Hawking radiation during black hole evaporation

Illustrative normalized curves, not observational data.

The unitary curve is expected to decrease after the Page time. Its precise shape depends on the model; the endpoint assumes complete unitary evaporation.

These developments provide substantial theoretical evidence that black hole information can be preserved, but they do not yet establish a complete experimentally verified account of black hole evaporation.

Question 2 — Summary table

Physical concept	Principal equation	Physical meaning
General relativity	$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$	Curvature and energy
Gravity	Geodesic equation	Motion in curved spacetime
Gravitational wave	$\Box \bar{h}_{\mu\nu} = 0$	Propagating metric perturbation
Quantum gravitational field	$\hat{h}_{\mu\nu}$	Graviton excitations in perturbation theory

Black hole	$r_s = 2GM/c^2$	Event horizon
Black hole entropy	$S_{BH} = k_B A/(4\ell_P^2)$	Entropy proportional to area
Black hole temperature	$T_H = \hbar c^3/(8\pi GMk_B)$	Hawking temperature
Free quantum field	$(\Box_g - m^2)\phi = 0$	Quantum field dynamics
Quantum field entropy	$S = -k_B \operatorname{Tr}\rho \ln \rho$	Thermal or reduced-state entropy
String microstates	$S = k_B \ln d(E)$	Statistical interpretation of entropy
Black hole evaporation	$dM/dt \propto -M^{-2}$	Hawking radiation and mass loss
Holographic principle	$Z_{\text{grav}} = Z_{\text{CFT}}$	Boundary/bulk dual description
Island formula	$S(R) = \min \operatorname{ext} S_{\text{gen}}$	Fine-grained radiation entropy

Question 3. Interconnections between mathematics and physics

We now arrive at the central question:

How can mathematical entropy, quantum wave functions, and information theory help us understand gravitational geometry and quantum gravity?

I will explain five important connections.

1. Probability theory → Quantum information → Black hole entropy

The first connection begins with the mathematical definition of entropy.

Step 1. Classical probability

$$S_{\text{Gibbs}} = -k_B \sum_i p_i \ln p_i.$$

Step 2. Quantum probability

Replace the probability distribution by a density operator:

$$\rho = \sum_i p_i |i\rangle\langle i|.$$

Then

$$S_{\text{vN}} = -k_B \operatorname{Tr}(\rho \ln \rho).$$

Step 3. Black hole microstates

If a black hole possesses Ω_{BH} equally probable microscopic states, then

$$S_{\text{micro}} = k_B \ln \Omega_{BH}.$$

The Bekenstein–Hawking formula suggests

$$\ln \Omega_{BH} = \frac{A}{4\ell_P^2}.$$

Therefore, formally,

$$\boxed{\Omega_{BH} \sim \exp\left(\frac{A}{4\ell_P^2}\right)}.$$

The striking result is that the number of states grows exponentially with the horizon area, rather than with the spatial volume.

This motivates the holographic principle.

2. Differential geometry → General relativity → Thermodynamics

The second connection is between spacetime curvature and entropy.

Step 1. Geometry

$$(M, g_{\mu\nu}) \longrightarrow R_{\mu\nu} \longrightarrow G_{\mu\nu}.$$

Step 2. Gravity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

for vanishing cosmological constant.

Step 3. Horizon geometry

A black hole solution possesses a horizon with area A.

Step 4. Entropy

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}.$$

The surprising point is that a quantity defined by differential geometry, A, becomes a thermodynamic quantity, S.

In Jacobson's thermodynamic approach, the Einstein equation can be related to the local Clausius relation

$$\boxed{\delta Q = T\, dS}$$

under assumptions involving local causal horizons, area entropy, and local equilibrium.

This suggests that Einstein's equation may admit a thermodynamic interpretation, although it does not prove that spacetime is fundamentally thermodynamic.

3. Hilbert spaces → Entanglement entropy → Spacetime geometry

The third connection is particularly important for holographic quantum gravity.

Let

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

The reduced state is

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB}.$$

Its entropy is

$$S_A = -k_B\, \mathrm{Tr}(\rho_A \ln \rho_A).$$

In the appropriate holographic setting,

$$\boxed{\frac{S_A}{k_B} = \frac{\mathrm{Area}(\gamma_A)}{4G_N\hbar}}$$

with $c = 1$.

This is the Ryu–Takayanagi relation. DOI

The equation relates two apparently different mathematical objects:

- The left side is calculated from a density operator on a Hilbert space.
- The right side is calculated from the geometry of a surface in a gravitational spacetime.

Thus, quantum entanglement and geometry are quantitatively connected.

4. Quantum field theory → Hawking radiation → Information conservation

The fourth connection involves quantum field theory in curved spacetime.

Consider two descriptions of quantum field modes, related through a Bogoliubov transformation:

$$b_i = \sum_j \left(\alpha_{ij} a_j + \beta_{ij} a_j^\dagger \right).$$

The coefficients β_{ij} describe mixing between positive- and negative-frequency modes.

A vacuum defined by one observer or mode decomposition can contain particles according to another.

In black hole formation, this mechanism leads to Hawking radiation.

The resulting question is whether the outgoing radiation retains the quantum information associated with the initial state.

The Page curve and island calculations connect the problem to entanglement entropy and quantum extremal surfaces.

5. String theory → Quantum microstates → Black hole entropy

String theory supplies a microscopic framework for certain black hole entropy calculations.

The mathematical progression is

$$\text{String states} \longrightarrow \text{D-brane configurations} \longrightarrow \text{Microstate counting} \longrightarrow S_{\text{micro}}.$$

For special supersymmetric black holes,

$$S_{\text{micro}} = S_{BH}$$

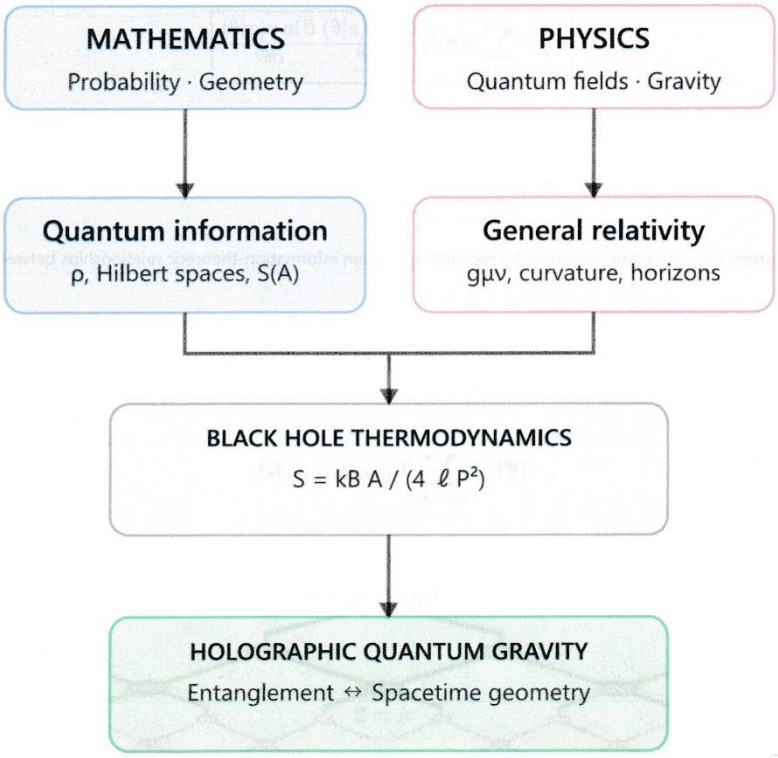
at leading order.

This provides a connection between quantum state counting and classical horizon geometry.

Question 3 — Comprehensive comparison table

Mathematics	Mathematical object or equation	Physics	Interconnection
Probability theory	p_i	Statistical mechanics	Probability of microstates
Information theory	$H = -\sum p_i \log_2 p_i$	Black hole information	Information content
Statistical entropy	$S = k_B \ln \Omega$	Black hole thermodynamics	Counting microstates
Hilbert space	$\mathcal{H}$	Quantum mechanics	Space of quantum states
Wave function	$\psi \in \mathcal{H}$	Quantum fields and quantum gravity	Quantum-state description
Density operators	ρ	Quantum information	Mixed and reduced states
Entanglement entropy	$S_A = -k_B \operatorname{Tr} \rho_A \ln \rho_A$	Black holes and holography	Quantum correlations
Tensor analysis	$R_{\mu\nu}, g_{\mu\nu}$	General relativity	Spacetime curvature
Differential equations	$\Box h_{\mu\nu} = 0$ in a suitable gauge	Gravitational waves	Wave propagation
Differential geometry	Horizon area A	Black hole entropy	$S_{BH} \propto A$
Functional analysis	Field operators	Quantum field theory	Field quantization
Statistical mechanics	$Z = \operatorname{Tr} e^{-\beta H}$	Hawking radiation	Thermal quantum states
Algebraic geometry	Calabi–Yau manifolds	String compactification	Extra-dimensional geometry
Representation theory	Group representations	String and particle states	Classification of quantum excitations
Holographic entropy	$\operatorname{Area}(\gamma_A)/(4G_N)$	AdS/CFT	Boundary entropy and bulk geometry
Quantum extremal surfaces	$\operatorname{ext} S_{\text{gen}}$	Black hole evaporation	Fine-grained radiation entropy

An integrated mathematical map



A conceptual connection, not a completed unification

The deepest idea is that **quantum information may help determine the structure of spacetime itself**.

However, the precise mechanism by which general spacetime geometry emerges from microscopic quantum degrees of freedom remains an open problem.

Question 4. Future mathematics and physics beyond current theories

I suggest approaching this question by distinguishing between established mathematical frameworks, promising theoretical developments, and unresolved questions.

1. Quantum information geometry

The first promising direction is the geometry of quantum states.

A family of quantum states can be described by density operators

$$\rho(\theta^1, \ldots, \theta^n).$$

The parameters θ^i define a statistical manifold.

For a classical probability family, the Fisher information metric is

$$\boxed{g_{ij}^F = \sum_x p(x|\theta) \frac{\partial \ln p(x|\theta)}{\partial \theta^i} \frac{\partial \ln p(x|\theta)}{\partial \theta^j}}.$$

Quantum analogues include the quantum Fisher information metric and the Bures metric.

These structures suggest possible connections among statistical distinguishability, quantum entanglement, and geometric descriptions of spacetime.

Future research question: Can spacetime geometry be reconstructed from information-theoretic relationships between quantum states?

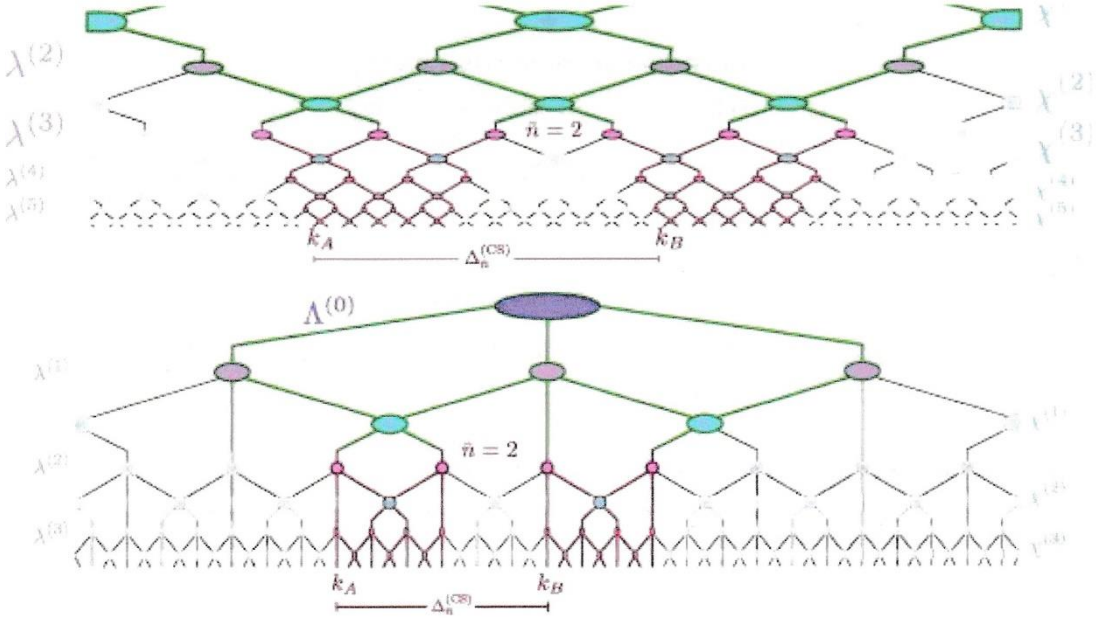
2. Tensor networks and emergent spacetime

Tensor networks represent complex quantum states through contractions of smaller tensors.

For example,

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} T_{i_1\cdots i_n} |i_1, \ldots, i_n\rangle.$$

A tensor network decomposes T into smaller interconnected tensors.



Certain tensor-network structures resemble spatial slices of holographic geometries.

They also connect naturally with quantum error-correcting codes, which provide mathematical models for how bulk information might be encoded redundantly in boundary degrees of freedom.

Future research question: Under what conditions can a network of entangled quantum degrees of freedom produce a smooth Lorentzian spacetime?

3. Quantum extremal surfaces and black hole information

A particularly active direction is the generalized entropy

$$\boxed{S_{\text{gen}} = \frac{k_B A}{4\ell_P^2} + S_{\text{bulk}}}$$

with appropriate renormalization.

The corresponding quantum extremal surface satisfies

$$\delta S_{\text{gen}} = 0.$$

The island formula extends this framework to the entropy of Hawking radiation.

These results have substantially improved the mathematical understanding of the black hole information paradox. scoap3-prod-ba...

Future research question: How does a complete microscopic quantum-gravity theory explain these semiclassical entropy formulas?

4. String theory, topology, and quantum geometry

String theory connects quantum gravity to several advanced areas of mathematics:

- Algebraic geometry and Calabi–Yau manifolds.
- Homology, cohomology, and K-theory.
- Mirror symmetry and moduli spaces.
- Conformal field theory and representation theory.
- Higher categories and derived geometry.

For example, a Calabi–Yau threefold X has a nontrivial cohomological structure described by groups

$$H^k(X, \mathbb{Z}).$$

The Hodge numbers

$$h^{p,q}(X) = \dim H^{p,q}(X)$$

encode important geometric properties of the compactification space.

Future research question: Which mathematical structures determine the physically realized spacetime and particle spectrum?

5. Loop quantum gravity and discrete geometry

Loop quantum gravity uses connection variables, holonomies, and representations of $SU(2)$.

The kinematical area operator has eigenvalues of the form

$$\boxed{A = 8\pi\gamma\ell_P^2 \sum_p \sqrt{j_p(j_p+1)}}$$

for punctures of a surface by spin-network edges, where j_p are spin labels and γ is the Barbero–Immirzi parameter.

This suggests a discrete quantum structure for geometric observables.

Future research question: How does a smooth classical spacetime emerge from quantum spin-network states, and can this approach reproduce black hole thermodynamics in a fully controlled manner?

6. Category theory and a unified mathematical framework

Category theory may provide a common language for relating mathematical structures used in quantum gravity.

For example, consider:

$$\mathcal{C}_{\mathrm{Geom}}$$

as a category of geometric structures and suitable morphisms, and

$$\mathcal{C}_{\mathrm{Quant}}$$

as a category of quantum systems and suitable transformations.

One can investigate mappings of the form

$$F : \mathcal{C}_{\mathrm{Geom}} \longrightarrow \mathcal{C}_{\mathrm{Quant}}.$$

In particular theories, such functorial structures can be made precise. However, there is no established universal functor that transforms all classical gravitational geometries into a complete quantum-gravity theory.

This approach is attractive because it emphasizes relations and transformations rather than isolated mathematical objects.

Future research question: Can higher category theory, operator algebras, and quantum information be combined into a mathematically rigorous framework for quantum spacetime?

Question 4 — Future study summary table

Future subject	Important mathematics	Physics objective	Status
Quantum information geometry	Fisher metrics, quantum relative entropy	Geometry from information	Active research
Tensor networks	Tensor algebra, graph theory	Emergent spacetime	Models and conjectures
Quantum error correction	Operator algebras, coding theory	Holographic reconstruction	Strong theoretical framework in suitable models
Quantum extremal surfaces	Variational calculus, generalized entropy	Black hole information	Established calculations in controlled models
String quantum gravity	Algebraic geometry, K-theory	Microscopic gravitational states	Candidate framework
Loop quantum gravity	$SU(2)$, spin networks	Quantum geometry	Candidate framework
Higher category theory	Categories, functors, higher morphisms	Structural unification	Mathematical research direction
Noncommutative geometry	Operator algebras, spectral triples	Quantum spacetime	Exploratory
Quantum cosmology	Functional analysis, constraint quantization	Quantum origin of the universe	Unresolved

Experimental quantum gravity	Quantum metrology, interferometry	Observable quantum gravitational effects	Developing research

Recommended study program

Because your main interest is the interconnection of mathematical structures and physical theories, I recommend the following sequence.

1. **Stage 1 — Entropy and probability**
   Study Shannon entropy, Gibbs entropy, density matrices, and von Neumann entropy. Derive the entropy of a two-level quantum system.
2. **Stage 2 — Quantum fields and curved spacetime**
   Study Hilbert spaces, free scalar fields, the Klein–Gordon equation, and Bogoliubov transformations.
3. **Stage 3 — Black hole thermodynamics**
   Derive the Schwarzschild horizon area, Bekenstein–Hawking entropy, Hawking temperature, and the black hole first law.
4. **Stage 4 — Entanglement and holography**
   Study AdS geometry, conformal field theory, the Ryu–Takayanagi formula, and quantum extremal surfaces.
5. **Stage 5 — Quantum gravity and new mathematics**
   Compare string theory, loop quantum gravity, tensor networks, operator algebras, and category-theoretic approaches.

Suggested academic readings

Book or paper	Main subject
Elements of Information Theory	Shannon entropy and information theory
Quantum Computation and Quantum Information	Density operators and quantum entropy
General Relativity	Spacetime geometry and black holes
Quantum Fields in Curved Space	Quantum fields and Hawking radiation
A First Course in String Theory	String theory foundations
Holographic Entanglement Entropy	Entanglement and holographic geometry

For a more advanced research-level study, the original Ryu–Takayanagi paper (2006) ↗ is particularly valuable.

Final synthesis: Can entropy become the foundation of quantum gravity?

We can summarize the entire discussion using four fundamental equations.

1. QUANTUM INFORMATION

$$S = -k_B \operatorname{Tr}(\rho \ln \rho)$$

2. CLASSICAL GRAVITY

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

3. BLACK HOLE THERMODYNAMICS

$$S_{BH} = \frac{k_B A}{4\ell_P^2}$$

4. HOLOGRAPHIC ENTANGLEMENT

$$S_A = \frac{k_B \,\mathrm{Area}(\gamma_A)}{4\ell_P^2}$$

Leading-order formula in an appropriate static Einstein-gravity holographic dual.

These equations express a profound relationship:

$$\boxed{\text{Quantum Information} \longleftrightarrow \text{Entropy} \longleftrightarrow \text{Geometry} \longleftrightarrow \text{Gravity}}$$

The arrows represent deep theoretical connections, not a proven equivalence among all four concepts.

My suggestion for your next study is to focus on one particularly important mathematical question:

Can Einstein's gravitational field equations be derived from quantum entanglement entropy?

There are important partial results, including derivations of linearized gravitational equations from entanglement relations in holographic settings. This makes the subject a natural bridge between differential geometry, quantum information, and quantum gravity.

scoap3-prod-ba...

Understanding those derivations step by step would be an excellent next stage in your study of the interconnections between mathematics and physics.

あなた

This is enough for today. I could have understood the interconnections well. Thank you so much. Eiji

ChatGPT

You're very welcome! I'm glad our discussion helped you understand the connections between mathematics, entropy, and quantum gravity.

It has been a pleasure exploring these fascinating subjects with you. I look forward to continuing our discussions about mathematics and physics.

Have a wonderful day, and see you next time!