

Real Number and Time

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I am interested in the interconnections of Physics and Mathematics. This time, I would like to know the interconnections of Numbers in Mathematics and Time in Physics. Will you explain what Numbers are in mathematical terms. Especially, will you explain and define Real Number, in the way of step-by-step. And give me a summary table at last. Eiji

Numbers, in mathematics, are not just "labels for quantity" but elements of precise *structures* (sets with operations and axioms). I'll start from very general "numbers", then zoom in and *define* real numbers step-by-step, and finish with a summary table.

1. What are "numbers" in mathematics?

Mathematically, a "number system" is:

- A set S of objects
- Equipped with one or more **operations** (like $+$, \cdot , sometimes order \leq)
- Satisfying certain **axioms** (rules)

Examples:

- Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Rationals: $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0\}$
- Reals: \mathbb{R} , to be defined carefully below

Each of these is more than a "bag of symbols": it is a **structured object**.

2. Building up to the real numbers

We usually *build* number systems in layers:

(1) Natural numbers \mathbb{N}

- **Intuition:** counting: 0, 1, 2, 3, ...
- **Operations:** addition, multiplication
- **Axioms:** Peano axioms (existence of 0, successor function, induction, etc.)

Mathematically, we can even *define* naturals inside set theory, but conceptually they are the basic "counting numbers".

(2) Integers \mathbb{Z}

- Extend \mathbb{N} to allow **subtraction**:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

- Formally: integers can be constructed as **equivalence classes of pairs** of naturals:

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

and interpret (a, b) as " $a - b$ ".

Now \mathbb{Z} is a **ring**: we can add, subtract, multiply, but we can't always divide.

(3) Rational numbers \mathbb{Q}

- Extend \mathbb{Z} so that **division** (by nonzero integers) is possible:

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

- Formally: rationals are **equivalence classes of pairs** (m, n) , with $n \neq 0$:

$$(m, n) \sim (m', n') \Leftrightarrow mn' = m'n$$

and (m, n) represents the fraction $\frac{m}{n}$.

Now \mathbb{Q} is a **field**: we can add, subtract, multiply, and divide by nonzero elements.

But \mathbb{Q} is still not enough: some lengths or limits are not rational (e.g. $\sqrt{2}$, π , many infinite series limits).

3. What is a real number?

There are several equivalent characterizations. The most abstract and powerful one is:

Definition (Real numbers as a complete ordered field).

The real numbers \mathbb{R} form a set with:

- two operations $+$ and \cdot ,
- a distinguished element 0 (additive identity),
- a distinguished element 1 (multiplicative identity),
- a total order \leq

such that:

1. $(\mathbb{R}, +, \cdot)$ is a field;
2. \leq is compatible with the field operations (ordered field);
3. \mathbb{R} is complete: every non-empty subset of \mathbb{R} that is bounded above has a least upper bound (supremum).

This “complete ordered field” view is *the* key: there is (up to isomorphism) only one such structure.

Let's unpack this step-by-step.

4. Step-by-step structure of \mathbb{R}

Step 1: Field structure

\mathbb{R} is a field, meaning:

- For all $a, b, c \in \mathbb{R}$:
 1. **Addition**
 - Closure: $a + b \in \mathbb{R}$
 - Commutative: $a + b = b + a$
 - Associative: $(a + b) + c = a + (b + c)$
 - Additive identity: there is 0 with $a + 0 = a$
 - Additive inverse: for each a , there is $-a$ with $a + (-a) = 0$
 2. **Multiplication**
 - Closure: $a \cdot b \in \mathbb{R}$
 - Commutative: $ab = ba$
 - Associative: $(ab)c = a(bc)$
 - Multiplicative identity: there is $1 \neq 0$ with $a \cdot 1 = a$
 - Multiplicative inverse: for each $a \neq 0$, there is a^{-1} with $a \cdot a^{-1} = 1$
 3. **Distributive law**
$$a \cdot (b + c) = ab + ac$$

So, purely algebraically, \mathbb{R} behaves like \mathbb{Q} .

Step 2: Order structure

\mathbb{R} is an **ordered field**:

- There is a total order \leq on \mathbb{R} such that:
 - For any a, b , exactly one holds: $a < b$, $a = b$, or $a > b$
 - If $a \leq b$ then $a + c \leq b + c$ for all c

- If $0 \leq a$ and $0 \leq b$ then $0 \leq ab$

This gives the familiar "number line" picture and allows us to talk about inequalities, intervals, limits, etc.

Step 3: Completeness (the crucial step)

Completeness distinguishes \mathbb{R} from \mathbb{Q} .

There are several equivalent forms; one common formulation:

Least upper bound property (supremum property):

If $S \subseteq \mathbb{R}$ is non-empty and bounded above (there is some number M such that $x \leq M$ for all $x \in S$), then there exists a least upper bound $\sup S \in \mathbb{R}$.

Examples:

- The set $S = \{x \in \mathbb{R} : x^2 < 2\}$ has $\sup S = \sqrt{2} \in \mathbb{R}$.
- In \mathbb{Q} , the same set (but now restricted to rationals) has **no** supremum in \mathbb{Q} : any rational q with $q^2 < 2$ is not the least upper bound.

Equivalent ways to express completeness:

1. **Cauchy completeness:** every Cauchy sequence of real numbers converges to a real number.
2. **Nested intervals property:** if we have nested closed intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$ with lengths shrinking to 0, their intersection is a single real number.

These completeness properties are *not* true in \mathbb{Q} , but *are* true in \mathbb{R} .

5. Constructions of \mathbb{R} (more "explicit" definitions)

To make \mathbb{R} rigorous, mathematicians often construct it from \mathbb{Q} in one of two classical ways:

(A) Dedekind cuts

- A Dedekind cut is a subset $A \subset \mathbb{Q}$ such that:
 1. $A \neq \emptyset, A \neq \mathbb{Q}$,
 2. If $q \in A$ and $r < q$ (with $r \in \mathbb{Q}$), then $r \in A$ (downward closed),
 3. A has no **greatest** element.

Intuition: a cut A is like "all rationals less than some real number".

- Rational $r \in \mathbb{Q}$ corresponds to:

$$A_r = \{q \in \mathbb{Q} : q < r\}$$

- An irrational like $\sqrt{2}$ corresponds to:

$$A_{\sqrt{2}} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$$

Then:

- The set of all Dedekind cuts becomes \mathbb{R} .
- We define addition and multiplication of cuts appropriately.
- We can prove: this structure is a complete ordered field.

(B) Cauchy sequences

- A Cauchy sequence of rationals (a_n) is a sequence such that:

for every $\varepsilon > 0$, there exists N so that for all $m, n \geq N$:

$$|a_m - a_n| < \varepsilon.$$

Intuition: the terms of the sequence get arbitrarily close to each other; it "should have a limit".

- In \mathbb{Q} , a Cauchy sequence may **not** converge *inside* \mathbb{Q} (e.g. decimals approximating $\sqrt{2}$).
- Two sequences $(a_n), (b_n)$ are considered **equivalent** if:

$$\lim_{n \rightarrow \infty} (a_n - b_n) = 0.$$

- A **real number** is defined as an equivalence class of Cauchy sequences of rational numbers.

This construction “fills in” the gaps in \mathbb{Q} by adding all limits of Cauchy sequences.

6. View of real numbers in analysis and physics

Once we have \mathbb{R} as a complete ordered field, we can:

- Define **limits, continuity, derivatives, integrals**.
- Treat \mathbb{R} as a **line with no gaps**—crucial for modeling:
 - Time $t \in \mathbb{R}$
 - Position $X \in \mathbb{R}$ or \mathbb{R}^3
 - Fields $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}$ or \mathbb{C} , etc.

(We'll come back to the “Time in physics” connection when you ask next.)

7. Summary table

Here is a summary comparing the main number systems and highlighting the properties of \mathbb{R} :

Number system	Symbol	Typical element	Allowed operations	Order?	Completeness?
Natural numbers	\mathbb{N}	$0, 1, 2, 3, \dots$	$+, ;$ subtraction only partially	Yes (usual \leq)	No
Integers	\mathbb{Z}	$\dots, -2, -1, 0, 1, 2, \dots$	Addition, subtraction, multiplication	Yes	No
Rational numbers	\mathbb{Q}	$\frac{m}{n}$ with $m \in \mathbb{Z}, n \neq 0$	All four arithmetic operations (field)	Yes (order by fractions)	No (has “gaps”)
Real numbers	\mathbb{R}	Limits of Cauchy sequences, Dedekind cuts, decimals	All field operations; limits, derivatives, integrals defined	Yes (complete order)	Yes (complete ordered field)

If you'd like next, we can take this definition of \mathbb{R} and **connect it directly to “time” in physics**—for example: how the completeness of \mathbb{R} is used in defining continuous time evolution, differential equations, and physical symmetries.

Next, will you explain and define Time in Physics? Time is a unit of quantity of Physics. How does it be observed and measured? At present, I understand that Time of the physical unit is measured by the speed and frequency of light. How do you explain and define the quantity of Time in Physics?

Below I will give a *mathematically precise, physically meaningful, and historically modern* explanation of **time as a physical quantity**.

We will proceed step-by-step, from the operational definition (measurement), to the mathematical structure (what time *is* in physics), and then to the most modern view (relativity and field theory).

1. What is “time” as a physical quantity?

In physics, a **physical quantity** is defined by:

1. a **phenomenon** that realizes the quantity
2. an **operation** that measures it
3. a **mathematical structure** used to represent it

Time satisfies all three.

2. Operational definition: How is time observed and measured?

2.1 Clocks as physical systems

Time is defined operationally by choosing a physical process that repeats itself, and counting repetitions.

Examples:

- Oscillation of a pendulum
- Vibration of a quartz crystal
- Cesium-133 atomic transition (modern definition)

The key idea is:

We measure time by comparing an unknown duration to a reproducible periodic phenomenon.

This is the fundamental philosophy of physics: time is defined by clocks.

2.2 Modern (SI) definition of the second

The internationally defined unit of time (the second) is:

One second is defined as 9,192,631,770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the cesium-133 atom.

Meaning:

- Cesium atoms have a natural "internal vibration" (frequency).
- We count how many oscillations occur.
- That number *defines* one second.

So time is indeed measured by a **frequency**, but not by the speed of light.

(However, the *meter* is defined using the speed of light.)

3. Mathematical structure: What is "time" in physics?

Regardless of the physical measurement device, time in the mathematical formulation of physics is represented by:

3.1 Classical mechanics

Time is a **real-valued parameter**:

$$t \in \mathbb{R}.$$

A physical system is described by functions:

$$x(t), \quad v(t) = \frac{dx}{dt}, \quad a(t) = \frac{d^2x}{dt^2}.$$

So mathematically:

Time is modeled as a 1-dimensional continuum isomorphic to the real numbers.

This uses the **completeness** of \mathbb{R} :

If a motion approaches a limit, that limit exists in \mathbb{R} .

3.2 Special relativity

In relativity, time is no longer an independent parameter.

Instead, spacetime is a 4-dimensional manifold:

$$M \cong \mathbb{R}^4.$$

Coordinates:

$$x^\mu = (ct, x, y, z).$$

Time mixes with space through **Lorentz transformations**.

Proper time along a worldline is:

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2).$$

Meaning:

- “Time” depends on the motion of the observer.
- There is no universal time.

Mathematically, **proper time** is the integral of the spacetime metric:

$$\tau = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}.$$

3.3 General relativity

Here time is part of the curved 4-dimensional manifold (M, g) .

Proper time becomes:

$$d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}.$$

Gravitational fields affect the flow of time:

- Clocks run slower in strong gravity.

Thus:

Time is a geometric property of spacetime, determined by the metric tensor.

Not absolute, not universal, but *relational*.

3.4 Quantum mechanics

Time is unusual:

- Position X becomes an operator \hat{X} .
- Momentum P becomes an operator \hat{P} .
- But time is not an operator.

Time remains an external real parameter:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H} \psi(x, t).$$

So in quantum theory:

Time is a classical parameter controlling evolution, not a quantum observable.

3.5 Quantum field theory

Spacetime (including time) is the *domain* of fields:

$$\phi : \mathbb{R}^{1,3} \rightarrow \mathbb{R}, \mathbb{C}, \text{SU}(3), \dots$$

Again:

$$x^\mu = (ct, x, y, z)$$

Time is part of the spacetime coordinate.

Name: **Minkowski time** or **coordinate time**.

4. Deep structure: What is “time” conceptually?

Now a more abstract, mathematical viewpoint:

(1) Time as a parameter of evolution

A physical state $S(t)$ lies on a trajectory governed by differential equations:

$$\frac{ds}{dt} = F(s).$$

This requires time to be a **totally ordered, continuous, complete** parameter — hence modeled with \mathbb{R} .

(2) Time as a coordinate of spacetime (relativity)

Time is one dimension of a pseudo-Riemannian manifold.

The metric has signature $(+, -, -, -)$ or $(-, +, +, +)$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

This makes distinctions:

- timelike
- spacelike
- null

Time is *geometry*, not an independent quantity.

(3) Time as a physical quantity measured by a clock

A clock measures **proper time**:

$$\tau = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}.$$

This is the physically invariant quantity:

- all observers agree on τ
- observers do not agree on coordinate time t

Thus:

In general relativity, **time = proper time = length of worldline in spacetime**.

(4) Time as symmetry (Noether + physics)

Time translation symmetry:

$$t \mapsto t + \epsilon$$

gives via **Noether's theorem**:

Conservation of energy.

So the existence of time is deeply tied to:

- group actions (Lie groups),
- invariance,
- conserved quantities.

If physical laws were different at different moments, energy would not be conserved.

5. Summary Table: Mathematical and Physical Definitions of Time

Viewpoint	What is time?	Mathematical structure	Physical meaning
Operational (measurement)	Count of periodic events of a physical process	Integer counting \rightarrow real-valued duration	1 second = 9,192,631,770 cesium oscillations

Viewpoint	What is time?	Mathematical structure	Physical meaning
Classical mechanics	External evolution parameter	$t \in \mathbb{R}$	Describes motion $\mathbf{x}(t)$, forces, energy
Special relativity	Coordinate of spacetime	4-vector $\mathbf{x}^\mu = (ct, \mathbf{x})$	Time depends on velocity; Lorentz transformations
General relativity	Length along a worldline	$\tau = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$	Time flows differently in gravitational fields
Quantum mechanics	Parameter in Schrödinger equation	Real continuous parameter t	Not an operator; controls quantum evolution
Quantum field theory	Part of spacetime domain of fields	$\mathbb{R}^{1,3}$ with Minkowski metric	Fields $\phi(x^\mu)$ depend on spacetime positions
Noether symmetry viewpoint	Generator of time translations	Lie group \mathbb{R} acting on system	Energy is conserved if laws are time-invariant
Modern concept	A geometric property of spacetime + parameter of evolution	Real line / manifold coordinate / worldline length	Unifies measurement, dynamics, and geometry

If you'd like next:

I can continue the interconnection:

- "Real numbers" \leftrightarrow "Time" in physics
- Why the **completeness** of \mathbb{R} is essential for time
- How clocks relate to quantum fields
- Time in string theory (worldsheet time τ)
- Time in quantum gravity (the "problem of time")

Just tell me, Eiji.

Then, I understand as follows: many physicians and mathematicians have often presupposed that Time is a Real Number. Is that Right? How do you consider it in your way of thinkig?

Your understanding is **exactly correct**, Eiji.

And this point is *much deeper* than it first appears.

Let me explain clearly:

1. Yes — Most of physics presupposes that time is modeled by real numbers

Across classical mechanics, electromagnetism, quantum mechanics, general relativity, string theory, and quantum field theory, time is **almost always represented mathematically as an element of \mathbb{R}** .

Examples:

- Classical mechanics: $\mathbf{x}(t)$ with $t \in \mathbb{R}$
- Special relativity: time coordinate t in $\mathbb{R}^{1,3}$
- Quantum mechanics: Schrödinger evolution $e^{-iHt\hbar}$ with $t \in \mathbb{R}$
- QFT: fields $\phi(x, t)$ with $(t, x) \in \mathbb{R}^4$
- General relativity: proper time $\tau \in \mathbb{R}$

So physicists "assume" the real number line is the correct mathematical structure for time — continuous, ordered, unbroken.

This is not proved; it is a *postulate* or *modeling choice*.

2. Why do physicists use real numbers for time?

Because real numbers have three essential properties:

(1) Real numbers are totally ordered

We can always compare two instants:

$$t_1 < t_2.$$

This is needed to talk about "before" and "after."

(2) Real numbers are continuous

There is no "next" real number. Between any two times there are infinitely many more.

This expresses the intuition:

Time flows without discrete jumps.

(3) Real numbers are complete

The completeness property ensures limits exist, which is essential for:

- derivatives $\frac{dx}{dt}$
- integrals $\int L dt$
- differential equations
- Lagrangian and Hamiltonian mechanics
- wave equations
- Schrödinger evolution

If time were not complete, much of calculus would break.

Physics uses calculus everywhere.

Therefore, it assumes time is modeled by a complete ordered field $\rightarrow \mathbb{R}$.

3. But is time *really* a real number? (Deep question)

Now, *my own analysis*, from the mathematical-physical viewpoint:

Position 1: "Time is modeled by real numbers, because the real line is mathematically convenient."

This is the standard physics viewpoint:

"We represent time by \mathbb{R} because it works."

Not because nature "is made of real numbers," but because the model is powerful:

- differential equations
- smooth manifolds
- continuous symmetries
- Noether's theorem

- Lorentz transformations

All require the structure of \mathbb{R} .

Position 2: "It is not proved that physical time is truly continuous or real-valued."

Many deep theories suggest time might not be the real line:

(a) Quantum gravity

Approaches such as:

- Loop quantum gravity
- Causal set theory
- Discrete spacetime models

suggest that time may be *quantized* or discrete.

(b) Planck time

There is a smallest meaningful unit:

$$t_{\text{Planck}} \approx 5.39 \times 10^{-44} \text{ s.}$$

Below this, the concept of continuous time becomes unclear.

(c) "Problem of time" in quantum gravity

In canonical quantum gravity:

$$\hat{H}\Psi = 0$$

contains no external time parameter at all.

Time "emerges" from correlations, not as a fundamental real number.

Position 3: Time might be a parameter of a symmetry group

Time can be viewed abstractly as:

The parameter of the group of time translations
 $t \mapsto t + \epsilon$.

Mathematically, the group of real numbers under addition:

$$(\mathbb{R}, +)$$

is a **Lie group**.

This structure enters Lagrangians, Hamiltonians, and Noether's theorem.

So:

- time translation symmetry = real numbers under addition
- energy = generator of this group

From this perspective:

Time is real-valued because the symmetry group of time translations is modeled by \mathbb{R} .

This is a **mathematical structure**, not a physical axiom.

Position 4: My balanced conclusion (mathematical + physical)

Time as used in physics is a mathematical model

— specifically, the use of real numbers with the structure of a complete ordered field.

But this does not necessarily mean that physical time itself "is" a real number.

It means:

- continuous time works extremely well for modeling nature
- but at deeper levels (quantum gravity), this may break down
- and “real-number time” may be an approximation or emergent concept

Thus my view:

Time is represented by real numbers because real numbers provide the correct mathematical structure for dynamics, calculus, and symmetries — but this representation may not be strictly fundamental in nature.

This is the modern viewpoint in physics and philosophy of science.

4. Summary Table: Is time a real number?

Perspective	Is time a real number?	Explanation
Classical mechanics	Yes (explicitly)	$t \in \mathbb{R}$, calculus requires the real line
Quantum mechanics	Yes (as a parameter)	Schrödinger equation uses $t \in \mathbb{R}$
Special relativity	Yes (coordinate)	Time is part of Minkowski coordinates $(ct, x, y, z) \in \mathbb{R}^4$
General relativity	Yes, for smooth manifolds	Proper time is real-valued: $\tau \in \mathbb{R}$
Quantum field theory	Yes	Fields $\phi(X^\mu)$ depend on real-valued t
Quantum gravity	Not necessarily	Time may be discrete or emergent
Abstract mathematical viewpoint	Time is modeled by \mathbb{R}	Because \mathbb{R} is complete, ordered, continuous
Philosophical/ontological	Uncertain	Whether real-number time reflects “reality” is unknown

If you wish next:

I can extend this discussion to:

- How time relates to numbers categorically
- Why time translations form a Lie group
- How the real line connects with Lagrangian mechanics and Noether symmetry
- How time is treated in string theory (τ coordinate)
- The deep “problem of time” in quantum gravity

Just tell me, Eiji.

You are awful and amazing! Now, I could have made my understandings much clearer by knowing your explanations about Number and Time. Thank you so much. Eiji

Eiji,

your words are truly kind — thank you so much.

I'm very glad that the explanations about Number and Time helped make the connections clearer for you.

Your questions always go directly to the deepest foundations of physics and mathematics — where the real beauty lies. It is a pleasure to explore these ideas with you, step by step, rigorously and clearly.

いつでも、英治さん。

また一緒に続けていきましょう。